Higher Categories Learning Seminar

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These notes cover the contents of Section 4.3 in [Cis19], on the topic of final objects.

Goal. The section, largely speaking, aims to do the following:

(1) Give a characterization of final objects in the ∞-categorical setting.

(2) Show that final objects in ∞-categories admit a characterization in terms of 2-categorical data.

The contents have been divided into smaller pieces for the sake of legibility. Throughout, there have been occasional attempts to parallel results with similar ones for 1-categories.

Contents

0 Recollections

We begin by recalling some notions from previous sections in [Cis19], so we may reference them later. Furthermore, there is some intuition which may be valuable to have and which is not provided in *loc cit*.

0.1 Final maps

Definition 0.1. A map $u : A \to B$ of simplicial sets is *final* if for any morphism $p : B \to C$, the induced map

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ & {\scriptstyle p\circ u}\searrow \quad \swarrow{\scriptstyle p} & \\ & C & \end{array} \quad \text{in } \mathbf{sSet}/C$$

is a weak equivalence of the contravariant model structure over C.

This definition warrants some explaining. The contravariant model structure over C provides for us the homotopy theory of presheaves of spaces on C; in particular, by taking fibrant replacements, we can consider the triangle above as being a comparison between right fibrations. In this way, it is clear that one is making some statement about presheaves.

On the other hand, in the context of 1-categories, the category of presheaves on a category $\mathcal{C}$ can be thought of as the formal cocompletion of $\mathcal{C}$, where a presheaf represents the formal colimit over the diagram given by its category of elements. Classically, a functor $F : I \to J$ between 1-categories is *final* (or cofinal, depending on the author) if for any J-diagram $D : J \to \mathcal{C}$, the induced comparison map

$$\varinjlim(D \circ F) \to \varinjlim(D)$$

is an isomorphism.

Combining the above, we may think of Definition 0.1 in the following way: the map $(A, pu) \to (B, p)$ is analogous to the comparison map between colimits above, and u being final means it should be an equivalence for any p.

We summarize below the major properties we need about final maps.

Corollary 0.2. [Cis19, Cor. 4.1.9] *The following statements hold.*

(1) A monomorphism of simplicial sets is final if and only if it is right anodyne.

(2) A morphism of simplicial sets is final if and only if it factors into a right anodyne extension followed by a trivial fibration.

Furthermore, the class of final maps is exactly the smallest class C *of morphisms in* **sSet** *satisfying the following properties.*

(a) C *is closed under composition.*

(b) For any pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, *if* f *and* $g \circ f$ *are in* C *then so is* g.

(c) C *contains all right anodyne extensions.*

Remark 0.3. We note that by [Cis19, Prop. 4.1.11], an equivalence $u : A \to B$ of ∞-groupoids is automatically final. Indeed, the cited proposition says it suffices to check the definition of being final on a single right fibration $B \to X$, for which we may choose $X = \Delta^0$.

0.2 Mapping spaces

Definition 0.4. Consider two simplicial sets $(p_F : F \to X), (p_G : G \to X) \in \mathbf{sSet}/X$ over X. We define the *mapping space* $\mathrm{Map}_X(F,G) = \mathrm{Map}_X(p_F, p_G)$ by the pullback

$$\begin{array}{ccc} \mathrm{Map}_X(F,G) & \hookrightarrow & \underline{\mathrm{Hom}}(F,G) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (p_G)_*} \\ \Delta^0 & \xhookrightarrow{p_F} & \underline{\mathrm{Hom}}(F,X) \end{array}$$

where we note that an object in $\mathrm{Map}_X(F,G)$ thus corresponds to a commutative triangle

$$\begin{array}{ccc} F & \longrightarrow & G \\ & {\scriptstyle p_F}\searrow \quad \swarrow{\scriptstyle p_G} & \\ & X. & \end{array}$$

Remark 0.5. Observe that if $F = \Delta^0$ and $p_F = x \colon \Delta^0 \to X$, then $\mathrm{Map}_X(\{x\}, G)$ is given by the fiber

$$\begin{array}{ccc} p_G^{-1}(x) & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_G} \\ \Delta^0 & \xrightarrow{x} & X. \end{array}$$

Cisinski writes $G_x := p_G^{-1}(x)$.

Below, we give a summary of the basic properties we need about these mapping spaces.

Proposition 0.6. *Consider $F \to X$ and $F' \to X$ $\mathbf{sSet}/X$, and let $f : F \to F'$ be a morphism over X. Suppose $G \to X$ is a right fibration. Then the following statements hold.*

(1) [Cis19, 4.1.12] *The simplicial set $\mathrm{Map}_X(F,G)$ is a Kan complex.*

(2) [Cis19, Prop. 4.1.13] *If f is a monomorphism, then $f_* \colon \mathrm{Map}_X(F',G) \to \mathrm{Map}_X(F,G)$ is a Kan fibration.*

(3) [Cis19, Prop. 4.1.14] *If f is a weak equivalence of the contravariant model structure over X, then $f_* \colon \mathrm{Map}_X(F',G) \to \mathrm{Map}_X(F,G)$ is an equivalence of ∞-groupoids.*

We will also need the following fact about the contravariant model structure over X.

Theorem 0.7. [Cis19, Thm. 4.1.16] *Let $F \to X$ and $G \to X$ be right fibrations, and let $f : F \to G$ be a morphism over X. The following are equivalent.*

(1) f is a weak equivalence for the contravariant model structure over X.

(2) f is a fibrewise equivalence.

(3) For any $Y \to X$, the induced map

$$f_* \colon \mathrm{Map}_X(Y,F) \to \mathrm{Map}_X(Y,G)$$

is an equivalence of ∞-groupoids.

1 Final objects

1.1 The definition of final objects

In the context of 1-categories, an object $c \in C$ is final if it is a colimit for the identity functor $\mathbb{1}_C : C \to C$. More generally, c is final if and only if the colimit of any functor $D : C \to \mathcal{D}$ is given by $D(c)$, which is to say: the inclusion $\{c\} \hookrightarrow C$ is final.

Definition 1.1. Let $X \in \mathbf{sSet}$. An object $x \in X$ is *final* if $\Delta^0 \xrightarrow{x} X$ is final (see Definition 0.1).

Remark 1.2. [Cis19, Rmk. 4.3.2] Note that x is final if and only if $\Delta^0 \xrightarrow{x} X$ is right anodyne. Indeed, it is a monomorphism, so by Corollary 0.2, being final is equivalent to being right anodyne.

Proposition 1.3. [Cis19, Prop. 4.3.3] *Consider a morphism $f : X \to Y$ of simplicial sets, where $x \in X$ is final. Then f is final if and only if $f(x) \in Y$ is final.*

Proof. We use properties (a) and (b) from Corollary 0.2, and look at the composition

$$\Delta^0 \xrightarrow{x} X \xrightarrow{f} Y$$

representing $f(x)$. If f is final, then (a) yields that $f(x)$ is final. Conversely, if $f(x)$ is final, then f is final by (b). ∎

1.2 The pointed join & the Yoneda lemma

We have a monoidal structure on $\mathbf{sSet}$ provided by the join $-*- : \mathbf{sSet}\times\mathbf{sSet} \to \mathbf{sSet}$, which we recall preserves connected colimits in each variable. However, we will be interested in *pointed* simplicial sets $\mathbf{sSet}_*$, and given a pointed simplicial set (X, x) and simplicial set S, there is no natural way to turn $X * S$ into a pointed simplicial set. To remedy this, we introduce the pointed join $X *_x S$.

Definition 1.4. Let $(X, x) \in \mathbf{sSet}_*$, and let $S \in \mathbf{sSet}$. We define the simplicial set $X *_x S$ by the pushout

$$\begin{array}{ccc} \Delta^0 * S & \longrightarrow & \Delta^0 \\ {\scriptstyle x*1_S}\downarrow & \ulcorner & \downarrow{\scriptstyle x'} \\ X * S & \dashrightarrow & X *_x S. \end{array}$$

We regard $X *_x S$ as pointed by the object x'. This determines a functor $\mathbf{sSet}_* \times \mathbf{sSet} \to \mathbf{sSet}_*$.

Remark 1.5. The pointed join operation is associative in the following sense: given $X \in \mathbf{sSet}_*$ and $S, S' \in \mathbf{sSet}$, the associativity isomorphisms for $- * -$ induce a canonical isomorphism

$$(X *_x S) *_{x'} S' \cong X *_x (S * S').$$

To see this, look at the diagram

$$\begin{array}{ccccc} \Delta^0 * S * S' & \longrightarrow & \Delta^0 * S' & \longrightarrow & \Delta^0 \\ \downarrow{\scriptstyle x*1_S*1_{S'}} & \ulcorner & \downarrow{\scriptstyle x'*1_{S'}} & \ulcorner & \downarrow \\ X * S * S' & \longrightarrow & (X *_x S) * S' & \longrightarrow & (X *_x S) *_{x'} S' \end{array}$$

where we note that the left square is a pushout since $- * S'$ preserves connected colimits, and the right square is a pushout by definition. It follows that the outer square is a pushout.

Definition 1.6. We define the functor $C\colon \mathbf{sSet}_* \to \mathbf{sSet}_*$ by $C\colon (X,x) \mapsto X *_x \Delta^0$.

Remark 1.7. Given a pointed simplicial set (X,x), the pointed simplicial set $(C(X,x),x')$ can be thought of one obtained by forcing x to be final in X. We will see how to motivate this rigorously shortly, but on a heuristic level, what happens is the following: first, we form the join $X * \{x'\}$, wherein x' is now a final object. Next, in forming $C(X,x)$, we identify x with x'. This has the effect that any two morphisms $f,g : y \to x$ are necessarily identified with the unique morphism $y \to x'$, hence making x final.

Lemma 1.8. [Cis19, Lemma 4.3.5] *Let $(X,x) \in \mathbf{sSet}_*$, and consider the pointed simplicial set $(C(X,x),x')$. Then x' is a final object in $C(X,x)$.*

Proof. An object is final if and only if its representing map is right anodyne; furthermore, $\Delta^0 \xrightarrow{x'} C(X,x)$ is the pushout of

$$x * 1_{\Delta^0} : \Delta^0 * \Delta^0 = \Delta^1 \to X * \Delta^0.$$

It therefore suffices to show that $x * 1_{\Delta^0}$ is right anodyne, so we need to show it has the left lifting property with respect to right fibrations. By virtue of [Cis19, Prop. 4.1.2], an inner fibration $p : A \to B$ is a right fibration if and only if for any object $a \in A$, the induced map $A/a \to B/p(a)$ is a trivial fibration. Thus, we make use of the correspondence of lifting problems

$$\begin{array}{ccc} \Delta^0 * \Delta^0 & \longrightarrow & A \\ {\scriptstyle x*1_{\Delta^0}}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ X * \Delta^0 & \longrightarrow & B \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} \Delta^0 & \longrightarrow & A/a \\ {\scriptstyle x}\downarrow & \nearrow & \downarrow \\ X & \longrightarrow & B/p(a) \end{array}$$

provided by the adjunction transposition between joins and slices. Here, a is the codomain of arrow in A given by $\Delta^0 * \Delta^0 = \Delta^1 \to A$. As the right-most vertical arrow is a trivial fibration, the lift exists when the arrow to its left is a monomorphism, which it is. ■

Proposition 1.9. *We have an adjunction*

$$\mathbf{sSet}_* \underset{(Y,y)\mapsto(Y/y,1_y)}{\overset{C}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{sSet}_*$$

That is, there is a natural bijection between pointed maps $(C(X,x),x') \to (Y,y)$ and pointed maps $(X,x) \to (Y/y,1_y)$.

Proof sketch. In the simplicial set $X * \Delta^0$, let us denote the adjoined object by ∞. A pointed map

$$(C(X,x),x') = (X *_x \Delta^0, x') \to (Y,y)$$

is naturally the same thing as a map

$$X * \Delta^0 \to Y \quad \text{such that} \quad (x \to \infty) \mapsto 1_y.$$

This, in turn is the naturally same thing as a map

$$X \to Y/y \quad \text{such that} \quad x \mapsto 1_y.$$

This completes the proof. □

Proposition 1.10. [Cis19, Prop. 4.3.7] *Let $X \in \mathbf{sSet}$, and consider an object $x \in X$. If the canonical map $X/x \to X$ has a section $s\colon X \to X/x$ such that $s(x) = 1_x$, then x is final. If X is an ∞-category, then the converse also holds.*

Proof sketch. Note that s defines a pointed map $(X, x) \to (X/x, 1_x)$. By the previous proposition, we can transpose s to a map $r : (C(X, x), x') \to (X, x)$. As s is a pointed section of $X/x \to X$, r will be a pointed retraction[a] of the inclusion $X \hookrightarrow C(X, x)$. This implies that $x\colon \Delta^0 \to X$ is a retract of $x'\colon \Delta^0 \to C(X, x)$. Since these are monomorphisms, being final is equivalent to being right anodyne, so we conclude by noting that right anodyne extensions are closed under retracts.

For the final statement: if X is an ∞-category, then $X/x \to X$ is a right fibration, then since $x\colon \Delta^0 \to X$ is right anodyne (being final) the lifting problem

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{1_x} & X/x \\ {\scriptstyle x}\downarrow & \nearrow & \downarrow \\ X & = & X \end{array}$$

admits a solution. □

[a]Why?

Corollary 1.11. [Cis19, Cor. 4.3.8] *Let $X \in \mathbf{sSet}$, and consider an object $x \in X$. Then 1_x is final in X/x.*

Proof. The object Δ^0 is a monoid object in the monoidal category $(\mathbf{sSet}, *)$, with unit $\varnothing \to \Delta^0$ and multiplication the unique map $\Delta^0 * \Delta^0 = \Delta^1 \to \Delta^0$. We also observed that the pointed join $X *_x S$ defines an associative action, and one easily sees it is furthermore unital (by considering $S = \varnothing$).

Combining the above two facts, one observes that $C = (- *_- \Delta^0)$ inherits the structure of a monoid object in $\mathrm{Fun}(\mathbf{sSet}_*, \mathbf{sSet}_*)$; that is, it is a monad. By abstract nonsense, a right adjoint of a monad is a comonad (with the structure maps induced by transposing the ones for the left adjoint using the calculus of mates).

Let us write V for the right adjoint, given by $(Y, y) \mapsto (Y/y, 1_y)$. Then the unit $\mathbb{1} \Rightarrow C$ transposes to the counit map $\varepsilon\colon V \Rightarrow \mathbb{1}$, which is given simply by the canonical pointed map $X/x \to X$. On the other hand the multiplication $C^2 \Rightarrow C$ transposes to some inscrutable comultiplication map $\delta\colon V \Rightarrow V^2$, which satisfies counitality:

$$\begin{array}{ccc} V^2 & \xRightarrow{V\varepsilon} & V \\ {\scriptstyle \delta}\Uparrow & // & \\ V & & \end{array}$$

which tells us that the comultiplication provides a section of $V\varepsilon$. In other words, we have a pointed section of the pointed map

$$(X/x)/1_x \to X/x$$

so Proposition 1.10 tells us that x is final. ■

The above facts let us prove a version of the Yoneda lemma. Recall that the right fibration $X/x \to X$ is supposed to correspond to the representable functor $X(-, x)$. For a 1-category C, a special case of the Yoneda lemma is the statement that for $c, c' \in C$, one has a natural bijection

$$\widehat{C}(h_c, h_{c'}) \cong C(c, c')$$

where $\widehat{C} = \mathrm{Fun}(C, \mathbf{Set})$.

Theorem 1.12. [Cis19, Thm. 4.3.9] *Let X be an ∞-category, $x \in X$ some object. Then $(\Delta^0 \xrightarrow{x} X) \in$ $\mathbf{sSet}/X$ has a canonical fibrant replacement in the contravariant model structure over X, given by $X/x \to X$.*

In particular, if $y \in X$ is another object, then there is a canonical equivalence of groupoids

$$\mathrm{Map}_X(X/x, X/y) \to X(x, y).$$

Proof. We have seen that $1_x \colon \Delta^0 \to X/x$ is defines a final object, and therefore

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{1_x} & X/x \\ & {\scriptstyle x}\searrow \quad \swarrow & \\ & X & \end{array}$$

defines a weak equivalence in the contravariant model structure over X. Since $X/x \to X$ is a right fibration, X/x is fibrant and we have proven the first statement.

For the second, we apply the properties listed in Proposition 0.6. In particular, since $\Delta^0 \xrightarrow{1_x} X/x$ is a monomorphism,

$$\mathrm{Map}_X(\{x\}, X/y) \to \mathrm{Map}_X(X/x, X/y)$$

is a Kan fibration by (2). By the weak equivalence worked out above, (3) implies this map is an equivalence of ∞-groupoids, hence a trivial fibration. Now, by Remark 0.5, $\mathrm{Map}_X(\{x\}, X/y)$ is given by the fiber of $X/y \to X$ over x, which by [Cis19, Cor. 4.2.10] is canonically equivalent as an ∞-groupoid to $X(x, y)$. ∎

1.3 Characterizing final objects ∞-categorically

We now prove the following characterization of final objects in an ∞-category, originally due to Joyal.

Theorem 1.13. [Cis19, Thm. 4.3.11] *Let X be an ∞-category, and an object $\omega \in X$. Denote by $\pi \colon X/\omega \to X$ the canonical map. Then the following are equivalent.*

(1) ω is a final object in X.

(2) For any object $x \in X$, the ∞-groupoid $X(x, \omega)$ is contractible.

(3) $\pi \colon X/\omega \to X$ is a trivial fibration.

(4) $\pi : X/\omega \to X$ is an equivalence of ∞-categories.

(5) $\pi : X/\omega \to X$ has a section sending ω to 1_ω.

(6) Any morphism $u : \partial\Delta^n \to X$ for which $n > 0$ and $u(n) = \omega$ arises as the restriction of a morphism $\Delta^n \to X$.

Proof. We begin with some easy equivalences.

(1)⇔(5). This is Proposition 1.10.

(3)⇔(4). Since π is a fibration between fibrant objects of the Joyal model structure, it is a trivial fibration if and only if it is a weak categorical equivalence, i.e. an equivalence of ∞-categories.

(3)⇔(6). This follows by applying the identity

$$\partial\Delta^n = (\Delta^{n-1} * \varnothing) \cup (\partial\Delta^{n-1} * \Delta^0)$$

and the standard lifting problem transposition between joins and slices.

(2)⇔(3). Since π is a fibration between fibrant objects of the contravariant model structure over X, it is a trivial fibration if and only if it is a weak equivalence for this model structure. Applying the second statement in Theorem 0.7, as well as our Yoneda lemma (or by reapplying [Cis19, Cor. 4.2.10]), this is equivalent to $X(x,\omega) \to \Delta^0$ being an equivalence of ∞-groupoids, i.e. $X(x,\omega)$ is contractible.

For the final part of the proof, we use that (1)⇔(5), and split into two parts, proving equivalence with (3) in one direction using (1) and in the other using (5).

(3)⇒(1). A trivial fibration is automatically final (see Corollary 0.2), so the composition

$$\Delta^0 \xrightarrow{1_\omega} X/\omega \xrightarrow{\pi} X,$$

namely $\omega\colon \Delta^0 \to X$, is final by closure under composition and Corollary 1.11.

(5)⇒(3). We are given a section $s : X \to X/\omega$ which sends a final object ω to a final object 1_ω, which by Proposition 1.3 implies that s is final. Since s is monic and final, it is a right anodyne extension by Corollary 0.2. Now, this implies π is a weak equivalence of the contravariant model structure over X (use that all right anodyne extensions are weak equivalences by [Cis19, Prop. 2.4.25], and then apply the 2-out-of-3 property). Hence, π is a trivial fibration. ∎

1.4 Characterizing final objects 2-categorically

Theorem 1.13 lets us immediately come to some corollaries with remarkable implications. We begin with some fun and easy ones.

Corollary 1.14. [Cis19, Cor. 4.3.12] *Let X be an ∞-category, $\omega \in X$ an object. Then*

$$\omega \textit{ is final in } X \implies \omega \textit{ is final in } \mathrm{ho}(X).$$

Proof. By [Cis19, Prop. 3.7.2], we have

$$\pi_0 X(x,\omega) \cong \mathrm{ho}(X)(x,\omega).$$

Applying (2) in Theorem 1.13, we are done. ∎

Corollary 1.15. [Cis19, Cor. 4.3.13] *The final objects in an ∞-category X form an ∞-groupoid which is either empty or contractible.*

Proof. Let $K \subseteq X$ be the full subcategory of final objects. One may interpret (2) in Theorem 1.13 as saying that the unique map $K \to \Delta^0$ is fully faithful. Unless K is empty, it is also essentially surjective, hence an equivalence. ∎

Remark 1.16. Both of the above results seem like they should be a simple consequences, but with the definitions used here, one sees that they are certainly non-trivial (given all the work that has lead up to them). In other approaches, e.g. the one in [Lur09], they are significantly easier to prove (as one essentially takes Theorem 1.13(2) as the definition). On the other hand, the approach of [Cis19] has the benefit of making certain other arguments much more formal.

Corollary 1.17. [Cis19, Cor. 4.3.14] *Let ω be final in an ∞-category X. For any simplicial set A, the constant functor $c_\omega\colon A \to X$ with value ω is a final object of $\underline{\mathrm{Hom}}(A, X)$.*

Proof. For this one, we need to import a result about the compatibility between $\underline{\mathrm{Hom}}$ and slices, specifically [Cis19, Prop. 4.2.12], which says that

$$\underline{\mathrm{Hom}}(A, X/\omega) \to \underline{\mathrm{Hom}}(A, X)/c_\omega$$

is an equivalence of ∞-categories. Now, by (4) in Theorem 1.13, we thus have equivalences

$$\underline{\mathrm{Hom}}(A, X) \to \underline{\mathrm{Hom}}(A, X/\omega) \to \underline{\mathrm{Hom}}(A, X)/c_\omega$$

which also shows that c_ω is final for the same reason. ■

The really interesting consequence is the following theorem, characterizing final objects entirely at the level of some homotopy categories.

Lemma 1.18. *Let X be an ∞-category, and consider two objects $x, y \in X$. Then, for any $A \in$ **sSet**, we have a canonical equivalence of ∞-categories*

$$\underline{\mathrm{Hom}}(A, X(x, y)) \simeq \underline{\mathrm{Hom}}(A, X)(c_x, c_y).$$

Proof. The functor $\underline{\mathrm{Hom}}(A, -)$ has the following properties:

(a) It commutes with limits, since it is right adjoint to $\times$.

(b) It preserves Joyal fibrations by [Cis19, Cor. 3.6.4].

(c) It preserves equivalences of ∞-categories by [Cis19, Thm. 3.6.9] and trivial fibrations by [Cis19, Cor. 3.1.7].

Therefore, it preserves homotopy pullbacks of isofibrations between ∞-categories. As a consequence, the homotopy pullback (see [Cis19, Cor. 2.3.28]) below left

$$\begin{array}{ccc} X(x,y) & \longrightarrow & \Delta^0 \\ \downarrow & \lrcorner\,\mathrm{ho} & \downarrow{\scriptstyle x} \\ X/y & \longrightarrow & X \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \underline{\mathrm{Hom}}(A, X(x,y)) & \longrightarrow & \Delta^0 \\ \downarrow & \lrcorner\;\mathrm{ho} & \downarrow{\scriptstyle c_x} \\ \underline{\mathrm{Hom}}(A, X/y) & \longrightarrow & \underline{\mathrm{Hom}}(A, X) \end{array}$$

gives rise to the homotopy pullback above right. Owing to the fact that we have a canonical weak equivalence $\underline{\mathrm{Hom}}(A, X/y) \simeq \underline{\mathrm{Hom}}(A, X)/c_y$, this shows that

$$\underline{\mathrm{Hom}}(A, X(x, y)) \simeq \underline{\mathrm{Hom}}(A, X)(c_x, c_y)$$

as desired. ■

Theorem 1.19. [Cis19, Thm. 4.3.16] *Let X be an ∞-category, and $\omega \in X$ an object. Then the following are equivalent.*

(1) ω is a final object in X.

*(2) For any $A \in$ **sSet**, the constant functor c_ω is a final object in $\mathrm{ho}(\underline{\mathrm{Hom}}(A, X))$.*

(3) For any $A \cong N(E)$ where E is a finite partially ordered set, the constant functor c_ω is a final

object in $\mathrm{ho}(\underline{\mathrm{Hom}}(A, X))$.

Above, N denotes the nerve functor.

Proof. (1) $\Rightarrow$ (2). We know by the above corollaries that c_ω is final in $\underline{\mathrm{Hom}}(A, X)$ and hence final in the homotopy category.

(2) $\Rightarrow$ (3). This is trivial, as (3) is immediately a special case of (2).

(3) $\Rightarrow$ (1). We assume that $A = N(E)$ and that c_ω is final in $\mathrm{ho}(\underline{\mathrm{Hom}}(A, X))$. Applying Lemma 1.18, we have that

$$\{*\} = \mathrm{ho}(\underline{\mathrm{Hom}}(A, X))(c_x, c_\omega) \cong \pi_0(\underline{\mathrm{Hom}}(A, X)(c_x, c_\omega)) \cong \pi_0(\underline{\mathrm{Hom}}(A, X(x, \omega)))$$

so every map $A \to X(x, \omega)$ is homotopic to a constant map. Therefore, applying the below Lemma 1.20, we see that $X(x, \omega)$ is contractible for all $x \in X$, so ω is final. ■

Lemma 1.20. [Cis19, Lemma 4.3.15] *Let $X \in$ **sSet** and assume that for any finite poset E, any map $N(E) \to X$ is Δ^1-homotopic to a constant map. Then $X \to \Delta^0$ is a weak homotopy equivalence.*

Proof. The proof relies on [Cis19, Prop. 3.8.10], which says that for a pointed Kan complex (Y, y), there is a canonical bijection $\pi_0(\underline{\mathrm{Hom}}_*(\partial\Delta^{n+1}, Y)) \cong \pi_n(Y, y)$, where one points $\partial\Delta^{n+1}$ however one pleases. The idea is to apply this to the nice fibrant replacement $Y = \mathrm{Ex}^\infty(X)$ of X. The essential claim is as follows:

($\star$) Any map $\partial\Delta^n \to \mathrm{Ex}^\infty(X)$ is Δ^1-homotopic to a constant map.

To see that this is true, note that as $\partial\Delta^n$ has only finitely many non-degenerate simplices, it is compact as an object of **sSet**. Therefore, a map as above factors as

$$\partial\Delta^n \to \mathrm{Ex}^i(X) \to \mathrm{Ex}^\infty(X), \quad \text{for some } i > 0.$$

By adjunction, we obtain a map $\mathrm{Sd}^i(\partial\Delta^n) \to X$; applying [Cis19, Lemmas 3.1.25 & 3.1.26], $\mathrm{Sd}^i(\partial\Delta^n)$ is the nerve of a finite poset, so this map is Δ^1-homotopic to a constant map. One deduces that $\partial\Delta^n \to \mathrm{Ex}^i(X)$ is Δ^1-homotopic to a constant map too, which proves the claim.

Now, choosing $n = 0$, we see that X is non-empty. Choosing $n = 1$, we see that it is connected, i.e. $\pi_0(X) = \{*\}$. From this, $\pi_n(X, x)$ does not depend on x, and applying the general case of ($\star$) we see that $\pi_n(X, x)$ is trivial for all n. Therefore, the map $\mathrm{Ex}^\infty(X) \to \Delta^0$ is a weak homotopy equivalence, from which we conclude that $X \to \Delta^0$ is a weak homotopy equivalence. ■

Remark 1.21. We take this moment to advertise the approach towards ∞-categories given by Emily Riehl & Dominic Verity in [RV22].

One may form an $(\infty, 2)$-*category* $\mathfrak{QCat}$ whose objects are ∞-categories and whose mapping ∞-categories are given by $\underline{\mathrm{Hom}}(-, -)$. This is an example of an ∞-*cosmos* in the sense of [RV22]. One may further form the *homotopy 2-category* $\mathfrak{hQCat}$ of $\mathfrak{QCat}$ by letting the mapping categories be given by the homotopy categories $\mathrm{ho}(\underline{\mathrm{Hom}}(-, -))$.

The thesis of [RV22] is that much of the basic theory of ∞-categories can be developed essentially synthetically in the context of an ∞-cosmos $\mathfrak{K}$, and particularly, that many concepts depend only on the homotopy 2-category $\mathfrak{hK}$. The above theorem is an example of this concept: it tells us that final objects in ∞-categories are detected on the level of mapping categories in $\mathfrak{hQCat}$.

Cisinski proves a more general version of this later in Chapter 6 of [Cis19]; in particular, he proves that adjoints depend only on 2-categorical data in the homotopy 2-category $\mathfrak{hQCat}$. This generalizes the case of final objects, as an object $\Delta^0 \to X$ of an ∞-category is a final object if and only if it is right adjoint to the unique map $X \to \Delta^0$.

One may develop a great deal of basic ∞-category theory in terms of the interplay between an ∞-cosmos $\mathfrak{K}$ and its homotopy 2-category $\mathfrak{h}\mathfrak{K}$, and this is done systematically in [RV22]. Indeed, at once towards the beginning of the book, they define adjoints in terms of the existence of unit and counit natural transformations satisfying the triangle identities in $\mathfrak{h}\mathfrak{K}$, and use this to define such things as initial/terminal objects. One can similarly define what it means for an ∞-category to admit all (co)limits of a given shape, though some more care is required when one is only concerned with the existence of the (co)limit of a particular diagram.

While the theory of ∞-categories in ∞-cosmoi presents a number of technical issues, it allows a model independent *mostly* synthetic approach to proofs; furthermore, this setting occasionally allows for the category theory in ∞-category theory to shine through somewhat more clearly, as many proofs end up being very similar to those for 1-categories. The downside is that one must additionally show that what one has proven agrees with the standard notions in one's prefered model, and that many *explicit* constructions become impossible; for example, an ∞-cosmos may not even have an object representing a category of ∞-groupoids, which leads to formulations of the Yoneda lemma becoming somewhat more tricky. In part, these issues are inevitable, as there are more "exotic" ∞-cosmoi which one should not think of as modeling $(\infty, 1)$-categories, but rather $(\infty, 1)$-categorical aspects of e.g. (∞, n)-categories.

1.5 An odd end which is only relevant much later

From doing a search through [Cis19], the following proposition is only used twice; in one of those times, it is merely a suggestion. Therefore, we have deferred it to this subsection at the end.

Proposition 1.22. [Cis19, Prop. 4.3.10] *Let X be an ∞-category, and ω an object in X. Suppose there is a natural transformation $a : 1_X \Rightarrow c_\omega$ from the identity of X to the contant functor with value ω, and suppose that component arrow $a_\omega : \omega \to \omega$ is homotopy equivalent to the identity, i.e. $[a_\omega] = [1_\omega]$ in $\mathrm{ho}(X)$. Then ω is final.*

Proof. By definition, the natural transformation determines a homotopy

$$h : \Delta^1 \times X \to X.$$

The fundamental claim is the following:

(⋆) We may replace h by a homotopy $h' : \Delta^1 \times X \to X$ such that h' restricted to $\Delta^1 \times \{\omega\}$ is the identity 1_ω.

By the assumption that $[a_\omega] = [1_\omega]$, we have a commutative triangle $\mu : \Delta^2 \to X$ of the form

$$\begin{array}{ccccc} & & \omega & & \\ & {\scriptstyle 1_\omega}\nearrow & \Downarrow\mu & \searrow{\scriptstyle a_\omega} & \\ \omega & & \xrightarrow[1_\omega]{} & & \omega \end{array}$$

and so we may consider the map

$$\tilde{h} : \Lambda^2_1 \times X \to X \quad \text{glued from} \quad \begin{cases} \Delta^{\{0,1\}} \times X \to X \text{ coming from } \Delta^1 \xrightarrow{1_X} \underline{\mathrm{Hom}}(X, X), \text{ and} \\ \Delta^{\{1,2\}} \times X \xrightarrow{h} X. \end{cases}$$

In other words, the map $\Lambda^2_1 \to \underline{\mathrm{Hom}}(X, X)$ described by the chain of natural transformations

$$1_X \overset{1}{\Rightarrow} 1_X \overset{a}{\Rightarrow} c_\omega.$$

In totality, the above data provides us with the following lifting problem

$$
\begin{array}{ccc}
\Delta^2 \times \{\omega\} \cup \Lambda^2_1 \times X & \xrightarrow{(\mu, \tilde{h})} & X \\
\big\downarrow & \overset{k}{\nearrow} & \big\downarrow \\
\Delta^2 \times X & \longrightarrow & \Delta^0
\end{array}
$$

which admits a solution since the left map is inner anodyne and X is an ∞-category. Setting $h' = k|_{\Delta^{\{0,2\}} \times X}$ yields the desired homotopy, proving ($\star$).

Transposing h', we get a natural transformation $a' \colon \Delta^1 \to \underline{\mathrm{Hom}}(X, X)$ whose component at ω is 1_ω. Transposing the opposite way, we have a functor $H' \colon X \to \underline{\mathrm{Hom}}(\Delta^1, X)$ such that $H'(\omega) = 1_\omega$. In fact, since $a' \colon 1_X \Rightarrow c_\omega$, the functor H' factors through

$$H' \colon X \to X//\omega \subseteq \underline{\mathrm{Hom}}(\Delta^1, X)$$

and defines a pointed section of the canonical projection $X//\omega \to X$. The canonical equivalence

$$X/\omega \to X//\omega$$

of [Cis19, Prop. 4.2.9] sends the final object of X/ω to 1_ω, and as equivalences of ∞-groupoids are final (see Remark 0.3), this means 1_ω is final in $X//\omega$. Finally, we have the retraction

$$
\begin{array}{ccccc}
\Delta^0 & = & \Delta^0 & = & \Delta^0 \\
\omega \big\downarrow & & \big\downarrow 1_\omega & & \big\downarrow \omega \\
X & \xrightarrow{H'} & X//\omega & \longrightarrow & X
\end{array}
$$

where the middle arrow is right anodyne (since it is a monic and final map), hence ω is final. ■

References

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