Higher Categories Learning Seminar

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These notes cover the contents of Section 4.3 in [Cis19], on the topic of final objects.

Goal. The section, largely speaking, aims to do the following:

- (1) Give a characterization of final objects in the ∞ -categorical setting.
- (2) Show that final objects in ∞ -categories admit a characterization in terms of 2-categorical data.

The contents have been divided into smaller pieces for the sake of legibility. Throughout, there have been occasional attempts to parallel results with similar ones for 1-categories.

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0 Recollections

We begin by recalling some notions from previous sections in [Cis19], so we may reference them later. Furthermore, there is some intuition which may be valuable to have and which is not provided in *loc cit*.

0.1 Final maps

Definition 0.1. A map $u : A \to B$ of simplicial sets is *final* if for any morphism $p : B \to C$, the induced map

$$A \xrightarrow{u} B$$

$$p \circ u \bigvee_{C} p$$
in sSet/C

is a weak equivalence of the contravariant model structure over *C*.

This definition warrants some explaining. The contravariant model structure over *C* provides for us the homotopy theory of presheaves of spaces on *C*; in particular, by taking fibrant replacements, we can consider the triangle above as being a comparison between right fibrations. In this way, it is clear that one is making some statement about presheaves.

On the other hand, in the context of 1-categories, the category of presheaves on a category *C* can be thought of as the formal cocompletion of *C*, where a presheaf represents the formal colimit over the diagram given by its category of elements. Classically, a functor $F : I \rightarrow J$ between 1-categories is *final* (or cofinal, depending on the author) if for any *J*-diagram *D* : $J \rightarrow C$, the induced comparison map

$$\lim(D \circ F) \to \lim(D)$$

is an isomorphism.

Combining the above, we may think of Definition 0.1 in the following way: the map $(A, pu) \rightarrow (B, p)$ is analogous to the comparison map between colimits above, and *u* being final means it should be an equivalence for any *p*.

We summarize below the major properties we need about final maps.

Corollary 0.2. [Cis19, Cor. 4.1.9] The following statements hold.

- (1) A monomorphism of simplicial sets is final if and only if it is right anodyne.
- (2) A morphism of simplicial sets is final if and only if it factors into a right anodyne extension followed by a trivial fibration.

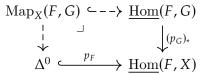
Furthermore, the class of final maps is exactly the smallest class C *of morphisms in* **sSet** *satisfying the following properties.*

- (*a*) C is closed under composition.
- (b) For any pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, if f and $g \circ f$ are in C then so is g.
- (c) C contains all right anodyne extensions.

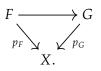
Remark 0.3. We note that by [Cis19, Prop. 4.1.11], an equivalence $u : A \to B$ of ∞ -groupoids is automatically final. Indeed, the cited proposition says it suffices to check the definition of being final on a single right fibration $B \to X$, for which we may choose $X = \Delta^0$.

0.2 Mapping spaces

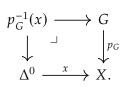
Definition 0.4. Consider two simplicial sets $(p_F : F \to X)$, $(p_G : G \to X) \in \mathbf{sSet}/X$ over X. We define the *mapping space* $Map_X(F, G) = Map_X(p_F, p_G)$ by the pullback



where we note that an object in $Map_X(F, G)$ thus corresponds to a commutative triangle



Remark 0.5. Observe that if $F = \Delta^0$ and $p_F = x : \Delta^0 \to X$, then $Map_X(\{x\}, G)$ is given by the fiber



Cisinski writes $G_x := p_G^{-1}(x)$.

Below, we give a summary of the basic properties we need about these mapping spaces.

Proposition 0.6. Consider $F \to X$ and $F' \to X$ sSet/X, and let $f : F \to F'$ be a morphism over X. Suppose $G \to X$ is a right fibration. Then the following statements hold.

- (1) [Cis19, 4.1.12] The simplicial set $Map_X(F, G)$ is a Kan complex.
- (2) [Cis19, Prop. 4.1.13] If f is a monomorphism, then $f_* \colon \operatorname{Map}_X(F', G) \to \operatorname{Map}_X(F, G)$ is a Kan fibration.
- (3) [Cis19, Prop. 4.1.14] If f is a weak equivalence of the contravariant model structure over X, then $f_* : \operatorname{Map}_X(F', G) \to \operatorname{Map}_X(F, G)$ is an equivalence of ∞ -groupoids.

We will also need the following fact about the contravariant model structure over *X*.

Theorem 0.7. [Cis19, Thm. 4.1.16] Let $F \to X$ and $G \to X$ be right fibrations, and let $f : F \to G$ be a morphism over X. The following are equivalent.

(1) *f* is a weak equivalence for the contravariant model structure over X.

- (2) *f* is a fibrewise equivalence.
- (3) For any $Y \rightarrow X$, the induced map

 $f_*: \operatorname{Map}_X(Y, F) \to \operatorname{Map}_X(Y, G)$

is an equivalence of ∞ -groupoids.

1 Final objects

1.1 The definition of final objects

In the context of 1-categories, an object $c \in C$ is final if it is a colimit for the identity functor $\mathbb{1}_C : C \to C$. More generally, c is final if and only if the colimit of any functor $D : C \to D$ is given by D(c), which is to say: the inclusion $\{c\} \hookrightarrow C$ is final.

Definition 1.1. Let $X \in$ sSet. An object $x \in X$ is *final* if $\Delta^0 \xrightarrow{x} X$ is final (see Definition 0.1).

Remark 1.2. [Cis19, Rmk. 4.3.2] Note that *x* is final if and only if $\Delta^0 \xrightarrow{x} X$ is right anodyne. Indeed, it is a monomorphism, so by Corollary 0.2, being final is equivalent to being right anodyne.

Proposition 1.3. [Cis19, Prop. 4.3.3] Consider a morphism $f : X \to Y$ of simplicial sets, where $x \in X$ is final. Then f is final if and only if $f(x) \in Y$ is final.

Proof. We use properties (a) and (b) from Corollary 0.2, and look at the composition

$$\Delta^0 \xrightarrow{x} X \xrightarrow{f} Y$$

representing f(x). If f is final, then (a) yields that f(x) is final. Conversely, if f(x) is final, then f is final by (b).

1.2 The pointed join & the Yoneda lemma

We have a monoidal structure on sSet provided by the join -*-: sSet×sSet \rightarrow sSet, which we recall preserves connected colimits in each variable. However, we will be interested in *pointed* simplicial sets sSet_{*}, and given a pointed simplicial set (*X*, *x*) and simplicial set *S*, there is no natural way to turn *X* * *S* into a pointed simplicial set. To remedy this, we introduce the pointed join $X *_x S$.

Definition 1.4. Let $(X, x) \in \mathbf{sSet}_*$, and let $S \in \mathbf{sSet}$. We define the simplicial set $X *_x S$ by the pushout

We regard $X *_x S$ as pointed by the object x'. This determines a functor $sSet_* \times sSet \rightarrow sSet_*$.

Remark 1.5. The pointed join operation is associative in the following sense: given $X \in \mathbf{sSet}_*$ and $S, S' \in \mathbf{sSet}$, the associativity isomorphisms for -*- induce a canonical isomorphism

$$(X *_{x} S) *_{x'} S' \cong X *_{x} (S * S').$$

To see this, look at the diagram

$$\begin{array}{cccc} \Delta^{0} * S * S' & \longrightarrow & \Delta^{0} * S' & \longrightarrow & \Delta^{0} \\ & \downarrow^{x * 1_{S} * 1_{S'}} & & \downarrow^{x' * 1_{S'}} & \downarrow \\ & X * S * S' & \longrightarrow & (X *_{x} S) * S' & \longrightarrow & (X *_{x} S) *_{x'} S' \end{array}$$

where we note that the left square is a pushout since -*S' preserves connected colimits, and the right square is a pushout by definition. It follows that the outer square is a pushout.

Definition 1.6. We define the functor $C : \mathbf{sSet}_* \to \mathbf{sSet}_*$ by $C : (X, x) \mapsto X *_x \Delta^0$.

Remark 1.7. Given a pointed simplicial set (X, x), the pointed simplicial set (C(X, x), x') can be thought of one obtained by forcing x to be final in X. We will see how to motivate this rigorously shortly, but on a heuristic level, what happens is the following: first, we form the join $X * \{x'\}$, wherein x' is now a final object. Next, in forming C(X, x), we identify x with x'. This has the effect that any two morphisms $f, g : y \to x$ are necessarily identified with the unique morphism $y \to x'$, hence making x final.

Lemma 1.8. [Cis19, Lemma 4.3.5] Let $(X, x) \in \mathbf{sSet}_*$, and consider the pointed simplicial set (C(X, x), x'). Then x' is a final object in C(X, x).

Proof. An object is final if and only if its representing map is right anodyne; furthermore, $\Delta^0 \xrightarrow{x'} C(X, x)$ is the pushout of

$$x * 1_{\Lambda^0} \colon \Delta^0 * \Delta^0 = \Delta^1 \longrightarrow X * \Delta^0.$$

It therefore suffices to show that $x * 1_{\Delta^0}$ is right anodyne, so we need to show it has the left lifting property with respect to right fibrations. By virtue of [Cis19, Prop. 4.1.2], an inner fibration $p : A \rightarrow B$ is a right fibration if and only if for any object $a \in A$, the induced map $A/a \rightarrow B/p(a)$ is a trivial fibration. Thus, we make use of the correspondence of lifting problems

$$\begin{array}{cccc} \Delta^{0} * \Delta^{0} & \longrightarrow & A & & \Delta^{0} & \longrightarrow & A/a \\ x * 1_{\Delta^{0}} \downarrow & & & \downarrow p & \longleftrightarrow & x \downarrow & & \downarrow \\ X * \Delta^{0} & \longrightarrow & B & & X & \longrightarrow & B/p(a) \end{array}$$

provided by the adjunction transposition between joins and slices. Here, *a* is the codomain of arrow in *A* given by $\Delta^0 * \Delta^0 = \Delta^1 \rightarrow A$. As the right-most vertical arrow is a trivial fibration, the lift exists when the arrow to its left is a monomorphism, which it is.

Proposition 1.9. We have an adjunction

$$sSet_* \underbrace{\downarrow}_{(Y,y)\mapsto (Y/y,1_y)}^C sSet_*$$

That is, there is a natural bijection between pointed maps $(C(X, x), x') \rightarrow (Y, y)$ and pointed maps $(X, x) \rightarrow (Y/y, 1_y)$.

Proof sketch. In the simplicial set $X * \Delta^0$, let us denote the adjoined object by ∞ . A pointed map

$$(C(X, x), x') = (X *_x \Delta^0, x') \to (Y, y)$$

is naturally the same thing as a map

 $X * \Delta^0 \to Y \quad \text{such that} \quad (x \to \infty) \mapsto 1_y.$

This, in turn is the naturally same thing as a map

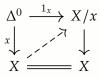
$$X \to Y/y$$
 such that $x \mapsto 1_y$

This completes the proof.

Proposition 1.10. [Cis19, Prop. 4.3.7] Let $X \in$ **sSet**, and consider an object $x \in X$. If the canonical map $X/x \to X$ has a section $s : X \to X/x$ such that $s(x) = 1_x$, then x is final. If X is an ∞ -category, then the converse also holds.

Proof sketch. Note that *s* defines a pointed map $(X, x) \rightarrow (X/x, 1_x)$. By the previous proposition, we can transpose *s* to a map $r : (C(X, x), x') \rightarrow (X, x)$. As *s* is a pointed section of $X/x \rightarrow X$, *r* will be a pointed retraction^{*a*} of the inclusion $X \rightarrow C(X, x)$. This implies that $x : \Delta^0 \rightarrow X$ is a retract of $x' : \Delta^0 \rightarrow C(X, x)$. Since these are monomorphisms, being final is equivalent to being right anodyne, so we conclude by noting that right anodyne extensions are closed under retracts.

For the final statement: if *X* is an ∞ -category, then $X/x \to X$ is a right fibration, then since $x \colon \Delta^0 \to X$ is right anodyne (being final) the lifting problem



admits a solution.

^aWhy?

Corollary 1.11. [Cis19, Cor. 4.3.8] Let $X \in \mathbf{sSet}$, and consider an object $x \in X$. Then 1_x is final in X/x.

Proof. The object Δ^0 is a monoid object in the monoidal category (sSet, *), with unit $\emptyset \to \Delta^0$ and multiplication the unique map $\Delta^0 * \Delta^0 = \Delta^1 \to \Delta^0$. We also observed that the pointed join $X *_x S$ defines an associative action, and one easily sees it is furthermore unital (by considering $S = \emptyset$).

Combining the above two facts, one observes that $C = (-*_{-} \Delta^{0})$ inherits the structure of a monoid object in Fun(sSet_{*}, sSet_{*}); that is, it is a monad. By abstract nonsense, a right adjoint of a monad is a comonad (with the structure maps induced by transposing the ones for the left adjoint using the calculus of mates).

Let us write *V* for the right adjoint, given by $(Y, y) \mapsto (Y/y, 1_y)$. Then the unit $1 \Rightarrow C$ transposes to the counit map $\varepsilon : V \Rightarrow 1$, which is given simply by the canonical pointed map $X/x \to X$. On the other hand the multiplication $C^2 \Rightarrow C$ transposes to some inscrutable comultiplication map $\delta : V \Rightarrow V^2$, which satisfies counitality:

$$V^2 \xrightarrow{V\varepsilon} V$$

which tells us that the comultiplication provides a section of $V\varepsilon$. In other words, we have a pointed section of the pointed map

 $(X/x)/1_x \to X/x$

so Proposition 1.10 tells us that *x* is final.

The above facts let us prove a version of the Yoneda lemma. Recall that the right fibration $X/x \rightarrow X$ is supposed to correspond to the representable functor X(-, x). For a 1-category C, a special case of the Yoneda lemma is the statement that for $c, c' \in C$, one has a natural

bijection

$$\widehat{C}(h_c, h_{c'}) \cong C(c, c')$$

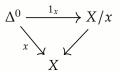
where $\widehat{C} = \operatorname{Fun}(C, \operatorname{Set})$.

Theorem 1.12. [Cis19, Thm. 4.3.9] Let X be an ∞ -category, $x \in X$ some object. Then $(\Delta^0 \xrightarrow{x} X) \in \mathbf{sSet}/X$ has a canonical fibrant replacement in the contravariant model structure over X, given by $X/x \to X$.

In particular, if $y \in X$ is another object, then there is a canonical equivalence of groupoids

$$\operatorname{Map}_X(X/x, X/y) \to X(x, y).$$

Proof. We have seen that $1_x \colon \Delta^0 \to X/x$ is defines a final object, and therefore



defines a weak equivalence in the contravariant model structure over X. Since $X/x \rightarrow X$ is a right fibration, X/x is fibrant and we have proven the first statement.

For the second, we apply the properties listed in Proposition 0.6. In particular, since $\Delta^0 \xrightarrow{1_x} X/x$ is a monomorphism,

$$\operatorname{Map}_X(\{x\}, X/y) \to \operatorname{Map}_X(X/x, X/y)$$

is a Kan fibration by (2). By the weak equivalence worked out above, (3) implies this map is an equivalence of ∞ -groupoids, hence a trivial fibration. Now, by Remark 0.5, $Map_X(\{x\}, X/y)$ is given by the fiber of $X/y \to X$ over x, which by [Cis19, Cor. 4.2.10] is canonically equivalent as an ∞ -groupoid to X(x, y).

1.3 Characterizing final objects ∞-categorically

We now prove the following characterization of final objects in an ∞ -category, originally due to Joyal.

Theorem 1.13. [Cis19, Thm. 4.3.11] Let X be an ∞ -category, and an object $\omega \in X$. Denote by $\pi: X/\omega \to X$ the canonical map. Then the following are equivalent.

- (1) ω is a final object in X.
- (2) For any object $x \in X$, the ∞ -groupoid $X(x, \omega)$ is contractible.
- (3) $\pi: X/\omega \to X$ is a trivial fibration.
- (4) $\pi: X/\omega \to X$ is an equivalence of ∞ -categories.
- (5) $\pi: X/\omega \to X$ has a section sending ω to 1_ω .
- (6) Any morphism $u : \partial \Delta^n \to X$ for which n > 0 and $u(n) = \omega$ arises as the restriction of a morphism $\Delta^n \to X$.

Proof. We begin with some easy equivalences. $(1) \Leftrightarrow (5)$. This is Proposition 1.10.

(3) \Leftrightarrow (4). Since π is a fibration between fibrant objects of the Joyal model structure, it is a trivial fibration if and only if it is a weak categorical equivalence, i.e. an equivalence of ∞ -categories.

 $(3) \Leftrightarrow (6)$. This follows by applying the identity

$$\partial \Delta^n = (\Delta^{n-1} * \emptyset) \cup (\partial \Delta^{n-1} * \Delta^0)$$

and the standard lifting problem transposition between joins and slices.

(2) \Leftrightarrow (3). Since π is a fibration between fibrant objects of the contravariant model structure over *X*, it is a trivial fibration if and only if it is a weak equivalence for this model structure. Applying the second statement in Theorem 0.7, as well as our Yoneda lemma (or by reapplying [Cis19, Cor. 4.2.10]), this is equivalent to $X(x, \omega) \rightarrow \Delta^0$ being an equivalence of ∞ -groupoids, i.e. $X(x, \omega)$ is contractible.

For the final part of the proof, we use that $(1) \Leftrightarrow (5)$, and split into two parts, proving equivalence with (3) in one direction using (1) and in the other using (5).

 $(3) \Rightarrow (1)$. A trivial fibration is automatically final (see Corollary 0.2), so the composition

 $\Delta^0 \xrightarrow{1_\omega} X/\omega \xrightarrow{\pi} X,$

namely $\omega : \Delta^0 \to X$, is final by closure under composition and Corollary 1.11.

 $(5)\Rightarrow(3)$. We are given a section $s : X \to X/\omega$ which sends a final object ω to a final object 1_{ω} , which by Proposition 1.3 implies that *s* is final. Since *s* is monic and final, it is a right anodyne extension by Corollary 0.2. Now, this implies π is a weak equivalence of the contravariant model structure over *X* (use that all right anodyne extensions are weak equivalences by [Cis19, Prop. 2.4.25], and then apply the 2-out-of-3 property). Hence, π is a trivial fibration.

1.4 Characterizing final objects 2-categorically

Theorem 1.13 lets us immediately come to some corollaries with remarkable implications. We begin with some fun and easy ones.

Corollary 1.14. [Cis19, Cor. 4.3.12] Let X be an ∞ -category, $\omega \in X$ an object. Then

 ω is final in $X \implies \omega$ is final in ho(X).

Proof. By [Cis19, Prop. 3.7.2], we have

$$\pi_0 X(x, \omega) \cong \operatorname{ho}(X)(x, \omega).$$

Applying (2) in Theorem 1.13, we are done.

Corollary 1.15. [Cis19, Cor. 4.3.13] *The final objects in an* ∞ *-category X form an* ∞ *-groupoid which is either empty or contractible.*

Proof. Let $K \subseteq X$ be the full subcategory of final objects. One may interpret (2) in Theorem 1.13 as saying that the unique map $K \to \Delta^0$ is fully faithful. Unless K is empty, it is also essentially surjective, hence an equivalence.

Remark 1.16. Both of the above results seem like they should be a simple consequences, but with the definitions used here, one sees that they are certainly non-trivial (given all the work that has lead up to them). In other approaches, e.g. the one in [Lur09], they are significantly easier to prove (as one essentially takes Theorem 1.13(2) as the definition). On the other

hand, the approach of [Cis19] has the benefit of making certain other arguments much more formal.

Corollary 1.17. [Cis19, Cor. 4.3.14] Let ω be final in an ∞ -category X. For any simplicial set A, the constant functor $c_{\omega} \colon A \to X$ with value ω is a final object of $\underline{\text{Hom}}(A, X)$.

Proof. For this one, we need to import a result about the compatibility between <u>Hom</u> and slices, specifically [Cis19, Prop. 4.2.12], which says that

 $\underline{\operatorname{Hom}}(A, X/\omega) \to \underline{\operatorname{Hom}}(A, X)/c_{\omega}$

is an equivalence of ∞ -categories. Now, by (4) in Theorem 1.13, we thus have equivalences

 $\operatorname{Hom}(A, X) \to \operatorname{Hom}(A, X/\omega) \to \operatorname{Hom}(A, X)/c_{\omega}$

which also shows that c_{ω} is final for the same reason.

The really interesting consequence is the following theorem, characterizing final objects entirely at the level of some homotopy categories.

Lemma 1.18. Let X be an ∞ -category, and consider two objects $x, y \in X$. Then, for any $A \in \mathbf{sSet}$, we have a canonical equivalence of ∞ -categories

$$\underline{\operatorname{Hom}}(A, X(x, y)) \simeq \underline{\operatorname{Hom}}(A, X)(c_x, c_y).$$

Proof. The functor $\underline{Hom}(A, -)$ has the following properties:

- (a) It commutes with limits, since it is right adjoint to \times .
- (b) It preserves Joyal fibrations by [Cis19, Cor. 3.6.4].
- (c) It preserves equivalences of ∞-categories by [Cis19, Thm. 3.6.9] and trivial fibrations by [Cis19, Cor. 3.1.7].

Therefore, it preserves homotopy pullbacks of isofibrations between ∞ -categories. As a consequence, the homotopy pullback (see [Cis19, Cor. 2.3.28]) below left

$$\begin{array}{cccc} X(x,y) & \longrightarrow & \Delta^{0} & & \underline{\operatorname{Hom}}(A,X(x,y)) & \longrightarrow & \Delta^{0} \\ & & \downarrow & \stackrel{\neg}{} & \operatorname{ho} & \downarrow^{x} & \rightsquigarrow & & \downarrow & \stackrel{\neg}{} & \operatorname{ho} & \downarrow^{c_{x}} \\ & X/y & \longrightarrow & X & & \underline{\operatorname{Hom}}(A,X/y) & \longrightarrow & \underline{\operatorname{Hom}}(A,X) \end{array}$$

gives rise to the homotopy pullback above right. Owing to the fact that we have a canonical weak equivalence $\underline{\text{Hom}}(A, X/y) \simeq \underline{\text{Hom}}(A, X)/c_y$, this shows that

$$\underline{\operatorname{Hom}}(A, X(x, y)) \simeq \underline{\operatorname{Hom}}(A, X)(c_x, c_y)$$

as desired.

Theorem 1.19. [Cis19, Thm. 4.3.16] Let X be an ∞ -category, and $\omega \in X$ an object. Then the following are equivalent.

- (1) ω is a final object in X.
- (2) For any $A \in \mathbf{sSet}$, the constant functor c_{ω} is a final object in ho(Hom(A, X)).
- (3) For any $A \cong N(E)$ where E is a finite partially ordered set, the constant functor c_{ω} is a final

object in ho(Hom(A, X)).

Above, N denotes the nerve functor.

Proof. (1) \Rightarrow (2). We know by the above corollaries that c_{ω} is final in <u>Hom</u>(A, X) and hence final in the homotopy category.

 $(2) \Rightarrow (3)$. This is trivial, as (3) is immediately a special case of (2).

(3) \Rightarrow (1). We assume that A = N(E) and that c_{ω} is final in ho($\underline{\text{Hom}}(A, X)$). Applying Lemma 1.18, we have that

 $\{*\} = \operatorname{ho}(\operatorname{Hom}(A, X))(c_x, c_\omega) \cong \pi_0(\operatorname{Hom}(A, X)(c_x, c_\omega)) \cong \pi_0(\operatorname{Hom}(A, X(x, \omega)))$

so every map $A \to X(x, \omega)$ is homotopic to a constant map. Therefore, applying the below Lemma 1.20, we see that $X(x, \omega)$ is contractible for all $x \in X$, so ω is final.

Lemma 1.20. [Cis19, Lemma 4.3.15] Let $X \in$ sSet and assume that for any finite poset E, any map $N(E) \to X$ is Δ^1 -homotopic to a constant map. Then $X \to \Delta^0$ is a weak homotopy equivalence.

Proof. The proof relies on [Cis19, Prop. 3.8.10], which says that for a pointed Kan complex (Y, y), there is a canonical bijection $\pi_0(\underline{\text{Hom}}_*(\partial \Delta^{n+1}, Y)) \cong \pi_n(Y, y)$, where one points $\partial \Delta^{n+1}$ however one pleases. The idea is to apply this to the nice fibrant replacement $Y = \text{Ex}^{\infty}(X)$ of *X*. The essential claim is as follows:

(*) Any map $\partial \Delta^n \to \operatorname{Ex}^{\infty}(X)$ is Δ^1 -homotopic to a constant map.

To see that this is true, note that as $\partial \Delta^n$ has only finitely many non-degenerate simplices, it is compact as an object of sSet. Therefore, a map as above factors as

$$\partial \Delta^n \to \operatorname{Ex}^i(X) \to \operatorname{Ex}^\infty(X), \text{ for some } i > 0.$$

By adjunction, we obtain a map $\operatorname{Sd}^{i}(\partial \Delta^{n}) \to X$; applying [Cis19, Lemmas 3.1.25 & 3.1.26], $\operatorname{Sd}^{i}(\partial \Delta^{n})$ is the nerve of a finite poset, so this map is Δ^{1} -homotopic to a constant map. One deduces that $\partial \Delta^{n} \to \operatorname{Ex}^{i}(X)$ is Δ^{1} -homotopic to a constant map too, which proves the claim.

Now, choosing n = 0, we see that X is non-empty. Choosing n = 1, we see that it is connected, i.e. $\pi_0(X) = \{*\}$. From this, $\pi_n(X, x)$ does not depend on x, and applying the general case of (\star) we see that $\pi_n(X, x)$ is trivial for all n. Therefore, the map $\text{Ex}^{\infty}(X) \to \Delta^0$ is a weak homotopy equivalence, from which we conclude that $X \to \Delta^0$ is a weak homotopy equivalence.

Remark 1.21. We take this moment to advertise the approach towards ∞-categories given by Emily Riehl & Dominic Verity in [RV22].

One may form an $(\infty, 2)$ -category QCat whose objects are ∞ -categories and whose mapping ∞ -categories are given by <u>Hom</u>(-, -). This is an example of an ∞ -cosmos in the sense of [RV22]. One may further form the *homotopy* 2-category hQCat of QCat by letting the mapping categories be given by the homotopy categories ho(Hom(-, -)).

The thesis of [RV22] is that much of the basic theory of ∞ -categories can be developed essentially synthetically in the context of an ∞ -cosmos \Re , and particularly, that many concepts depend only on the homotopy 2-category \Re . The above theorem is an example of this concept: it tells us that final objects in ∞ -categories are detected on the level of mapping categories in \Re Cat.

Cisinski proves a more general version of this later in Chapter 6 of [Cis19]; in particular, he proves that adjoints depend only on 2-categorical data in the homotopy 2-category \mathfrak{hQCat} . This generalizes the case of final objects, as an object $\Delta^0 \to X$ of an ∞ -category is a final object if and only if it is right adjoint to the unique map $X \to \Delta^0$.

One may develop a great deal of basic ∞ -category theory in terms of the interplay between an ∞ -cosmos \Re and its homotopy 2-category $\mathfrak{h}\mathfrak{R}$, and this is done systematically in [RV22]. Indeed, at once towards the beginning of the book, they define adjoints in terms of the existence of unit and counit natural transformations satisfying the triangle identities in $\mathfrak{h}\mathfrak{R}$, and use this to define such things as initial/terminal objects. One can similarly define what it means for an ∞ -category to admit all (co)limits of a given shape, though some more care is required when one is only concerned with the existence of the (co)limit of a particular diagram.

While the theory of ∞ -categories in ∞ -cosmoi presents a number of technical issues, it allows a model independent *mostly* synthetic approach to proofs; furthermore, this setting occasionally allows for the category theory in ∞ -category theory to shine through somewhat more clearly, as many proofs end up being very similar to those for 1-categories. The downside is that one must additionally show that what one has proven agrees with the standard notions in one's prefered model, and that many *explicit* constructions become impossible; for example, an ∞ -cosmos may not even have an object representing a category of ∞ -groupoids, which leads to formulations of the Yoneda lemma becoming somewhat more tricky. In part, these issues are inevitable, as there are more "exotic" ∞ -cosmoi which one should not think of as modeling (∞ , 1)-categories, but rather (∞ , 1)-categorical aspects of e.g. (∞ , *n*)-categories.

1.5 An odd end which is only relevant much later

From doing a search through [Cis19], the following proposition is only used twice; in one of those times, it is merely a suggestion. Therefore, we have deferred it to this subsection at the end.

Proposition 1.22. [Cis19, Prop. 4.3.10] Let X be an ∞ -category, and ω an object in X. Suppose there is a natural transformation $a : 1_X \Rightarrow c_\omega$ from the identity of X to the contant functor with value ω , and suppose that component arrow $a_\omega : \omega \to \omega$ is homotopy equivalent to the identity, i.e. $[a_\omega] = [1_\omega]$ in ho(X). Then ω is final.

Proof. By definition, the natural transformation determines a homotopy

$$h: \Delta^1 \times X \to X.$$

The fundamental claim is the following:

(*) We may replace *h* by a homotopy $h': \Delta^1 \times X \to X$ such that *h'* restricted to $\Delta^1 \times \{\omega\}$ is the identity 1_{ω} .

By the assumption that $[a_{\omega}] = [1_{\omega}]$, we have a commutative triangle $\mu: \Delta^2 \to X$ of the form

$$\omega \xrightarrow{1_{\omega}} \omega \xrightarrow{a_{\omega}} \omega$$

and so we may consider the map

$$\tilde{h}: \Lambda_1^2 \times X \to X$$
 glued from $\begin{cases} \Delta^{\{0,1\}} \times X \to X \text{ coming from } \Delta^1 \xrightarrow{1_X} \operatorname{Hom}(X, X), \text{ and} \\ \Delta^{\{1,2\}} \times X \xrightarrow{h} X. \end{cases}$

In other words, the map $\Lambda_1^2 \rightarrow \underline{Hom}(X, X)$ described by the chain of natural transformations

$$1_X \stackrel{1}{\Rightarrow} 1_X \stackrel{a}{\Rightarrow} c_\omega$$

In totality, the above data provides us with the following lifting problem

which admits a solution since the left map is inner anodyne and X is an ∞ -category. Setting $h' = k|_{\Delta^{\{0,2\}} \times X}$ yields the desired homotopy, proving (\star).

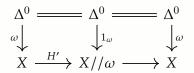
Transposing h', we get a natural transformation $a' : \Delta^1 \to \underline{\text{Hom}}(X, X)$ whose component at ω is 1_{ω} . Transposing the opposite way, we have a functor $H' : X \to \underline{\text{Hom}}(\Delta^1, X)$ such that $H'(\omega) = 1_{\omega}$. In fact, since $a' : 1_X \Rightarrow c_{\omega}$, the functor H' factors through

$$H': X \to X / \omega \subseteq \operatorname{Hom}(\Delta^1, X)$$

and defines a pointed section of the canonical projection $X//\omega \rightarrow X$. The canonical equivalence

$$X/\omega \rightarrow X//\omega$$

of [Cis19, Prop. 4.2.9] sends the final object of X/ω to 1_{ω} , and as equivalences of ∞ -groupoids are final (see Remark 0.3), this means 1_{ω} is final in $X//\omega$. Finally, we have the retraction



where the middle arrow is right anodyne (since it is a monic and final map), hence ω is final.

References

- [Cis19] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge studies in advanced mathematics. Cambridge University Press, 2019. ISBN: 978-1-108-47320-0.
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