t-STRUCTURES AND RECOLLEMENTS

CARL-FREDRIK LIDGREN

Abstract

We provide an introduction to the basic theory of triangulated categories and t-structures, and cover these in detail with a particular focus on *recollements,* roughly speaking gluings. Triangulated categories are a convenient setting for developing foundational results in homotopical algebra, and t-structures allow a refinement of this by imposing that objects have a kind of "grading" on them, encoded in certain (co)reflective subcategories. We prove a number of results on these topics, including a result of Hoshino–Kato–Miyachi stating that triangulated categories admitting small coproducts and a *silting object* also admit a t-structure whose heart is equivalent to a module category.

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Populärvetenskaplig Sammanfattning

Inom algebra finns det ett koncept som heter *kohomologi.* Från början kom det från topologi: till ett rum kan man associera en mängd algebraiska strukturer, nämligen *kohomologigrupperna* av rummet, som berättar om vissa egenskaper rummet har. Man märkte relativt snabbt att sådana strukturer kunde användas i fler situationer, och på så vis föddes ämnet *homologisk algebra.* Det man såg var att man kunde skapa en kohomologiteori inom alla så kallade *abelska kategorier.*

Inte långt efter att kohomologi hade definierats inom abelska kategorier insåg vissa matematiker att man behövde något mer flexibelt för mer krävande situationer. Anledningen är inte så svår att förstå: kohomologi kommer alltid från vissa följder

$$\cdots \to X^{i-1} \to X^i \to X^{i+1} \to \cdots$$

av morfismer (tänk: funktioner) mellan algebraiska objekt, där man beräknar kohomologin $\mathrm{H}^i(X^\bullet)$ i princip genom att kolla på skillnaden mellan avbildningen av $X^{i-1} \to X^i$ och det som $X^i \to X^{i+1}$ sänder till noll. Det är omedelbart uppenbart att man förlorar information genom att göra detta: man slänger bort en ganska stor del av följden.

Detta löstes genom att man bytte ut den initiala abelska kategorin med en ny kategori vars objekt är exakt sådana följder som ovan, fast där man betraktar två föjder som "samma" ungefär om de ger samma kohomologi. På så sätt glömmer man inte följden man började med. Denna metod har dock ett annat problem: den nya kategorin man använder är inte längre en abelsk kategori. Däremot har den liknande egenskaper som en abelsk kategori förrutom att de inte längre gäller "direkt" utan på ett mer avancerat sätt.

Den abstrakta strukturen som kommer fram i sammanhanget ovan kallas en *triangulerad kategori.* Denna uppsatts handlar om precis dessa kategorier, tillsammans med extra strukturer som man kan placera på dem, till exempel så kallade *t-strukturer,* som fungerar som ett abstrakt sätt att formulera hur man kommer ihåg följderna som diskuterades innan och konsekvenserna det har.

CONTENTS

1 INTRODUCTION

1.1 SOME HISTORY

This thesis concerns many tightly related topics centered around the concept of a t-structure on a triangulated category, and the main goal of the thesis is to present a reasonably self-contained exposition of these. Put broadly, triangulated categories are abstract settings in which one can do certain weakened constructions of homological algebra, and in particular they allow one to do some "elementary" version of homological algebra up to homotopy (so triangulated categories can be thought of as some homotopy-weakening of Abelian categories). This is perhaps surprisingly important even in studying ordinary homological algebra: in some sense, homological algebra is at its roots all about studying Abelian categories along with their *derived categories* (which we discuss in Section 4), however the latter crucially fail to avoid issues of homotopy—the derived category of an Abelian category is not an Abelian category, but it *is* a triangulated category. It may be useful to understand a bit of the history involved in order to see why one should care at all. For a much more detailed description of the history of homological algebra, on which we base our own, see the excellent [Wei99].

The modern notion of homological algebra has its roots in the book by Cartan & Eilenberg [CE56] from 1956, which itself was a response to a number of developments within topology—for example, the Ext-groups defined by Baer in 1934 [Bae34] were investigated by Eilenberg and MacLane in 1942 [EM42] in order to state a version of the universal coefficient theorem, and earlier than that Čech (in [Čec32]) had given a version of the universal coefficient theorem using what we would today recognize as Tor-groups. All of the work prior to the book of Cartan & Eilenberg was done purely in the context of Abelian groups and in terms of specific examples. These ideas were put into a more cohesive form in [CE56] under the name *derived functors,* which also allowed the authors to extend them to the context of modules over an arbitrary ring.

Around the same time, Leray (in [Ler46a; Ler46b], and [Ler50]) was developing the notion of a sheaf (most of his early work on which was done in a concentration camp), their cohomology and tools for computing said cohomology, namely spectral sequences (together with Koszul's work in [Kos47b; Kos47a]). Some of these methods, for example those of spectral sequences, were also later included in [CE56]. Still, all of these methods were contained to the world of modules over a ring.

This was the state of affairs until Grothendieck, in his 1957 "Tohoku paper" [Gro57], introduced a good notion of *Abelian categories* and detailed a method for doing homological algebra in such a framework, including generalizing the notion of derived functors. This allowed the techniques of homological algebra to be applied more directly to situations such as categories of sheaves on topological spaces, which were previously inaccessible except through ad-hoc methods.

For many purposes, the homological algebra of [Gro57] (or even [CE56] and other contemporary sources) was sufficient. However, for the work Grothendieck was doing in algebraic geometry, he needed a more flexible framework. In particular, in order to state and prove a duality theorem for sheaf cohomology, he needed to consider a homological algebra not just of "objects in an Abelian category" but of chain complexes of such objects. To elaborate, the methods available to him at the time essentially consisted of procedures one applies to individual objects, one at a time. What he needed was something which bundled together all these procedures into a single structure, and which enriched the individual objects to chain complexes of objects *up to cohomology* (more precisely, quasi-isomorphism).

To this end, Grothendieck sketched an idea about *derived categories,* which was later made into a precise notion by his student Jean-Louis Verdier in his thesis around 1967 (though only published in full in 1996 as [Ver96]). In his thesis, in order to prove results about derived categories, he introduced the general framework of *triangulated categories.*

Triangulated categories proved to be a very fruitful framework for doing quite a lot of foundational theory surrounding a homotopical version of homological algebra, in particular for having control over localizations, and giving a precise notion of what it means for a functor to be (co)homological (which allowed a rigorous way to describe how cohomological functors turn short exact sequences into long exact sequences). On the other hand, one finds that triangulated categories do not have enough structure and information in order to capture all the features of *derived* categories, which might be an issue given that triangulated categories were invented precisely to explain the structure of derived categories.

This situation was solved in [BBDG18] with the introduction of *t-structures.* A t-structure is additional data one puts on a triangulated category which essentially endows the objects of the category with a kind of "grading," or "chain complex structure," much like how the objects of a derived category consist of actual chain complexes. Once one has this additional data, it becomes possible to extract from it things like cohomology functors (which, appropriately, are examples of cohomological functors), and truncation functors (generalizing the concept of "cutting off" a chain complex at some point). Furthermore, it gives one the ability to define (t-)exact functors, namely (triangulated) functors which preserve this additional "grading." The benefit of this is that it provides a precise way in which some functors between derived categories preserve more or less structure.

On the other hand is another recurring topic in this thesis, namely recollements. Their name, translated from French, means something like "regluings" (or "patchings"), which suggests a picture that they describe how two triangulated categories are glued together to form a third. Roughly speaking, they are split short exact sequences of triangulated categories, and they do indeed lead to certain "gluing" phenomena which otherwise do not occur in general. Recollements are also interesting from a purely theoretical standpoint since they demonstrate very well how to operate the structure of a triangulated category.

1.2 STRUCTURE OF THE THESIS

At the end of this document, on page 122, is a table displaying some of the notation used throughout the thesis.

We begin, in Section 2, with a short introduction to some of the prerequisites regarding Abelian categories, in particular introducing chain complexes and their cohomology in an abstract setting. Notably, however, this section should not be taken as a complete description of all the prerequisites on this topic. The section is essentially entirely based on the exposition of Abelian categories found in [KS06], though in a less general setting.

In Section 3 we introduce triangulated categories, which will form the main setting of interest for most of the thesis. The basic theory is contained in Section 3.1, where we discuss the foundational definitions and their consequences. A notable result of interest is Proposition 3.15, which shows that the Hom functors in a triangulated category are *cohomological.* After this, in Section 3.2, we discuss some pertinent questions regarding uniqueness of certain constructions in triangulated categories, in particular Lemma 3.28 gives a criterion for when cones have a unique induced map.

Sections 3.3 and 3.4 are dedicated to building up the machinery required to define the Verdier quotient of a triangulated category by a null system (i.e. a triangulated subcategory closed under isomorphism), roughly speaking a coherent way to kill the objects of the Verdier quotient while still retaining a triangulated structure. The former is focused on the general theory of localization of categories, while the latter then specializes that to the context of triangulated categories. Amongst the important results of Section 3.3 are Proposition 3.42, which gives an explicit construction of the localization of a category by a suitably nice class of morphisms (called a multiplicative system), and Proposition 3.46, which says that the localization of an additive category (see Section 2.1) by a (left or right) multiplicative system of morphisms yields an additive category. This is then used in Section 3.4 to show that the Verdier quotient of a triangulated category is again a triangulated category, and that the canonical "projection" functor is triangulated; see Theorem 3.56. Another important result here is a characterization of when a morphism becomes invertible in the Verdier quotient (given in Corollary 3.67).

The last part of Section 3 is Section 3.5, which covers some of the theory of *recollements* (used later in Section 5.6). Notable results here are Theorems 3.75 and 3.87, which encapsulate very many of the formal categorical properties of recollements. The latter theorem also gives access to some non-trivial distinguished triangles; see Corollary 3.89. The references for Section 3 are primarily [KS06] and [Nee01], but also include [Kra22] and [MM92], particularly for Section 3.5.

Section 4 exists primarily to give some explicit non-trivial examples of the machinery of Section 3 "in the wild," namely derived categories of Abelian categories. The derived category is obtained by taking the category of chain complexes in an Abelian category, identifying morphisms under *homotopy,* then taking the Verdier quotient of this with respect to *acyclic complexes,* i.e. those whose cohomology is zero. In this way, one is considering a category of chain complexes "up to quasi-isomorphism," and this is made precise in Corollary 4.29. We present the fundamentals here in a reasonably complete way, though with some arguments only being sketched. The exposition here is largely based on [KS06], especially in Sections 4.2 and 4.3, although the proof of Theorem 4.28 is based on an analogous proof for topological spaces (as outlined in [Ram21]), and the proof of Lemma 4.26 was independently worked out. The proof of Corollary 4.29 is based on the outline in [Pap20].

In Section 4 one also sees the first motivations for t-structures, which are the subject of Section 5. There, we treat t-structures, and there are a number of central results of interest. In Section 5.1, we give the definition of a t-structure and construct one of the most important features they have, namely the truncation functors. In Section 5.2, we prove that the *heart* of a t-structure is an Abelian category, and following this we construct some *cohomology functors* associated to a t-structure in Section 5.3. These give a sequence of functors $\mathrm{H}^i : \mathcal{D} \to \mathcal{D}^\heartsuit$ from a triangulated category $\mathcal{D}$ with a t-structure into the heart $\mathcal{D}^\heartsuit$ of the t-structure, and in Section 5.5 we show that these functors are cohomological. In between, we also have an important result in Section 5.4 showing that *(Yoneda) extensions* in the heart $\mathcal{D}^\heartsuit$ can be computed using a Hom-functor; see Theorem 5.30. Essentially all of these results are used later in Section 6.

In Section 5.6, we relate recollements to t-structures. More precisely, we first show that given a suitably compatible "short exact sequence" (i.e. Verdier quotient sequence, in our terminology) of triangulated categories with t-structures, the t-structures on the smaller pieces determines the one on the bigger piece, and vice versa (see Propositions 5.42 and 5.43). We then show what could be considered the "main" result of the subsection, Theorem 5.45, which says that given a *recollement* where the smaller pieces have t-structures, this (uniquely) determines a t-structure on the bigger piece (i.e. the "gluing"). We end with giving two toy examples of applying this.

The main reference for Section 5 is [KS94], which covers essentially everything except the content of Section 5.4 and most of Section 5.6. For Section 5.4, the proof of Theorem 5.30 was worked out independently with some initial guiding direction from my advisor. The lion's share of what is in Section 5.6 (with the exception of the final part of Theorem 5.45, which is taken directly from the original source, [BBDG18]) was worked out independently based on the very brief outline in [GM03, p. 286, Ex. IV.4.2] and with some occasional tips from my advisor.

Section 6 covers just enough in order to use the previously developed theory to prove Theorem 6.21, which is due to Hoshino–Kato–Miyachi. This theorem is the focal point of the section, and it says that in any sufficiently nice triangulated category (one which admits a *silting object,* that is, a compact generator with certain Hom-properties) admits a similarly nice t-structure. In order to prove this, we define compact objects in Section 6.2 and prove a proposition about triangulated categories which are generated by some set of objects. In Section 6.3, we define "the" homotopy colimit of a certain kind of diagram, and use this to prove an "approximation" result, namely Theorem 6.16. These results are then combined to prove Theorem 6.21. The primary resource used in this section is [Jas21].

1.3 PREREQUISITES

It is somewhat difficult to estimate the precise required prerequisites for this thesis. The main necessity is a healthy amount of category theory and facts from homological algebra. As noted before, this thesis is very much not self-contained, and one does need a strong foundation in the aforementioned subjects. As a result, we take very many of the results from [KS06] for granted, particularly those of their Chapter 8, amongst other things. All such results are standard, and therefore should not be too hard to look up should it occur that a reference has been neglected. Outside of these, there are essentially no formal prerequisites other than those required by the previous two.

1.4 A NOTE ON FOUNDATIONS

There are various approaches to making the foundations of category theory precise, and in practice it is not of great importance which one is used. One approach is based on choosing three inaccessible cardinals, one corresponding to "small," another to some ordinary, default size, and a final for "large." To use this approach, one must then postulate the existence of such inaccessible cardinals. Another approach, which we take in this thesis (following [KS06], who follow [SGA4]), uses universes.

Definition 1.1. Let $\mathcal{U}$ be a set. We say $\mathcal{U}$ is a *(Grothendieck) universe* if it satisfies the following properties:

(i) $\varnothing \in \mathcal{U}$,

(ii) if $u \in \mathcal{U}$, then $u \subset \mathcal{U}$,

(iii) if $u \in \mathcal{U}$, then $\{u\} \in \mathcal{U}$,

(iv) if $u \in \mathcal{U}$, then the power set $\mathcal{P}(u) \in \mathcal{U}$,

(v) if $I \in \mathcal{U}$ and $u_i \in \mathcal{U}$ for all $i \in I$, then $\bigcup_{i \in I} u_i \in \mathcal{U}$, and

(vi) $\mathbb{N} \in \mathcal{U}$.

A universe is then essentially a set in which one can perform practically any set-theoretical operation of interest upon its elements, and still remain within the universe. We add the following axiom to Zermelo-Fraenkel set theory:

Axiom. Any set is contained in a universe. That is, for any set X, there exists a universe $\mathcal{U}$ such that $X \in \mathcal{U}$.

For the thesis, we fix some universe $\mathcal{U}$, and make the following definitions.

Definition 1.2. We say a set S is *small* if $S \in \mathcal{U}$. A category $\mathcal{C}$ is *locally small* if for all objects $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is small. We say $\mathcal{C}$ is *small* if it is locally small, and furthermore, the set of all objects in $\mathcal{C}$ is small.

By default one should assume that, unless otherwise specified, all sets and categories are small although it should be noted that this is usually not actually necessary.

2 ADDITIVE & ABELIAN CATEGORIES

2.1 (PRE-)ADDITIVE CATEGORIES

We want a natural place (i.e. category) in which to do homological algebra. At its most basic level, homological algebra occurs in the category of Abelian groups, **Ab**, and so we should aim to replicate features of this category. One of the most immediate properties is that if $A, B \in \mathbf{Ab}$ are Abelian groups, then $\mathrm{Hom}_{\mathbf{Ab}}(A, B)$ is also an Abelian group. In particular, for any two morphisms $f, g\colon A \to B$, we can define $(f+g)(a) = f(a)+g(a)$. Furthermore, this is well-behaved with respect to composition: for all composable morphisms f, g, h, $(f+g) \circ h = (f \circ h) + (g \circ h)$, and $h \circ (f+g) = (h \circ f) + (h \circ g)$. Notably, the composition map $\circ$ is $\mathbb{Z}$-bilinear. We may then pose the following definition:

Definition 2.1. A *pre-additive* category is a category $\mathcal{C}$ such that for each $A, B \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has the structure of an Abelian group, and for each $A, B, C \in \mathcal{C}$ the map

$$\circ\colon \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

is bilinear.

In the category of Abelian groups, finite products and coproducts agree. This is also always true in pre-additive categories.

Proposition 2.2. *Let $\mathcal{C}$ be a pre-additive category, and let $A_1, A_2 \in \mathcal{C}$.*

(a) Suppose the product $A_1 \times A_2$ exists, and let $p_k\colon A_1 \times A_2 \to A_k$ be the projections. Define the maps $i_k\colon A_k \to A_1 \times A_2$ to be the maps induced by $\mathrm{id}_{A_k}\colon A_k \to A_k$ and the zero map. Then

$$\mathrm{id}_{A_1 \times A_2} = (i_1 \circ p_1) + (i_2 \circ p_2).$$

(b) Let $B \in \mathcal{C}$ and assume there are maps $p_k\colon B \to A_k$, $i_k\colon A_k \to B$ such that

$$\mathrm{id}_B = (i_1 \circ p_1) + (i_2 \circ p_2) \quad \textit{and} \quad p_j \circ i_k = \begin{cases} \mathrm{id}_{A_k} & \textit{if } j = k, \\ 0 & \textit{if } j \neq k. \end{cases}$$

Then B together with (p_1, p_2) is a product of A_1 and A_2, and B together with (i_1, i_2) is a coproduct of A_1 and A_2.

Proof sketch. For a full proof, see [KS06, pp. 169–170, Lemma 8.2.3]. To prove *(a)*, note that

$$p_1 \circ ((i_1 \circ p_1) + (i_2 \circ p_2)) = (p_1 \circ i_1 \circ p_1) + (p_1 \circ i_2 \circ p_2) = p_1 + 0 \circ p_2 = p_1 \circ \mathrm{id}_{A_1 \times A_2}.$$

A symmetric calculation shows the same thing for p_2, and so by the universal property of the product we obtain the desired equality.

To prove *(b)*, first pick an arbitrary $Y \in \mathcal{C}$ and apply $\mathrm{Hom}_{\mathcal{C}}(Y, -)$. Since this commutes with products, this lets us assume that $\mathcal{C} = \mathbf{Ab}$. In particular, to see that B is a product of A_1 and A_2, it suffices to see that $\mathrm{Hom}_{\mathcal{C}}(Y, B)$ is a product of $\mathrm{Hom}_{\mathcal{C}}(Y, A_1)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, A_2)$ in the category **Ab**, functorially for all $Y \in \mathcal{C}$. The proposition can then be checked using explicit calculations, in particular by checking that the canonical maps $B \to A_1 \times A_2$ and $A_1 \sqcup A_2 \to B$ are isomorphisms. ☐

Corollary 2.3. *Let $\mathcal{C}$ be a pre-additive category, and let $A_1, A_2 \in \mathcal{C}$. Whenever their product exists, so does their coproduct. Furthermore, we have an isomorphism $r\colon A_1 \sqcup A_2 \to A_1 \times A_2$ induced by the maps i_k from above.*

Definition 2.4. Let $\mathcal{C}$ be a pre-additive category. When a product $A \times B$ exists, we denote it by $A \oplus B$ and call it the *direct sum.*

Above, one sees that products and coproducts agree for all finite, but non-empty, cases. The following gives us the empty case:

Lemma 2.5. *Let $\mathcal{C}$ be a pre-additive category, let $A, B, C \in \mathcal{C}$, and suppose we have morphisms $f \colon A \to B$, $g \colon B \to C$. Then $0 \circ f = 0$ and $g \circ 0 = 0$.*

Proof. By bilinearity of composition, we have that

$$0 \circ f + 0 \circ f = (0+0) \circ f = 0 \circ f \implies 0 \circ f = 0$$

and

$$g \circ 0 + g \circ 0 = g \circ (0+0) = g \circ 0 \implies g \circ 0 = 0.$$

This proves the lemma. ■

Proposition 2.6. *Let $\mathcal{C}$ be a pre-additive category. If an object $* \in \mathcal{C}$ is terminal, then it is initial. Conversely, if an object $\varnothing \in \mathcal{C}$ is initial, then it is terminal.*

Proof. Suppose $* \in \mathcal{C}$ is terminal. Then there is a unique map $* \to *$ which must also be the identity, so in particular $\mathrm{id}_* = 0 \in \mathrm{Hom}(*, *)$. Now let $A \in \mathcal{C}$ and suppose we have a morphism $f \colon * \to A$ (of which there exists at least one, since the hom-sets are Abelian groups). By Lemma 2.5, we then have

$$f = f \circ \mathrm{id}_* = f \circ 0 = 0 \implies f = 0.$$

Therefore, any map $* \to A$ is zero, so there is exactly one map $* \to A$ for each $A \in \mathcal{C}$, i.e. $*$ is inital. The case where $\varnothing$ is initial follows by a dual argument. ■

An interesting feature of pre-additive categories is that much of the additive structure is totally determined by the underlying category.

Proposition 2.7. *Let $\mathcal{C}$ be a pre-additive category, let $A, B \in \mathcal{C}$, and let $f, g \in \mathrm{Hom}(A, B)$. If the direct sums $A \oplus A$ and $B \oplus B$ exist, then the morphism $f + g \in \mathrm{Hom}(A, B)$ is given by the composition*

$$A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\nabla_B} B$$

where the first map is the diagonal map and the last map is the codiagonal, i.e. the map

$$B \oplus B \to B$$

induced by the universal property of the coproduct along with the identity map id_B.

Proof. Let $p_1^A, p_2^A \colon A \oplus A \to A$, $i_1^A, i_2^A \colon A \to A \oplus A$ be as before (and similarly define maps for B). Then it is easily computed that

$$p_1^A \circ (i_1^A + i_2^A) = (p_1^A \circ i_1^A) + (p_1^A \circ i_2^A) = \mathrm{id}_A = p_1^A \circ \Delta_A$$

and similarly for p_2^A, so $i_1^A + i_2^A = \Delta_A$. An essentially identical proof shows that $\nabla_B = p_1^B + p_2^B$. Therefore, we have that

$$\nabla_B \circ (f \oplus g) \circ i_1^A = f, \quad \nabla_B \circ (f \oplus g) \circ i_2^A = g,$$

and therefore

$$\nabla_B \circ (f \oplus g) \circ \Delta_A = (\nabla_B \circ (f \oplus g)) \circ (i_1^A + i_2^A) = f + g$$

as desired. ■

So we see that the additive structure on the Hom-sets in a pre-additive category is totally determined by the underlying category whenever that makes sense, i.e. when the product exists. Therefore, it is suggestive for an *additive* category to be a pre-additive category where such products always exists. We make the following definition:

Definition 2.8. An *additive* category is a pre-additive category such that all finite products (including the empty product) exist.

Remark 2.9. Alternatively, one can define an additive category as a category $\mathcal{C}$ such that

(a) there is a zero-object $0 \in \mathcal{C}$,
(b) for any $A, B \in \mathcal{C}$, the product $A \times B$ and the coproduct $A \sqcup B$ exist,
(c) for any $A, B \in \mathcal{C}$, the morphism $r\colon A \sqcup B \to A \times B$ induced by the maps $(\mathrm{id}_A, 0)\colon A \to A \times B$, $(0, \mathrm{id}_B)\colon B \to A \times B$ is an isomorphism, and
(d) for every $A \in \mathcal{C}$, there is some $a \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ such that the composition

$$A \xrightarrow{\Delta_A} A \times A \xrightarrow{(a,\mathrm{id}_A)} A \times A \xleftarrow{r} A \sqcup A \xrightarrow{\nabla_A} A$$

is the zero morphism.

It is then clear that a pre-additive category admitting all finite products satisfies all these properties (with the morphism a above being $-\mathrm{id}_A$). It is also not too hard to show the converse (see [KS06, p. 173, Thm. 8.2.14]). The final condition, (d), is essentially the existence of additive inverses. Intuitively, it is imposing the existence of a map $a\colon A \to A$ such that for all $x \in A$, $a(x) + x = 0$.

Definition 2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be (pre-)additive categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *additive* if for each $A, B \in \mathcal{C}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is a homomorphism of Abelian groups.

Since the underlying category of an additive category totally determine the structure, we should expect that additive functors too are characterized by a purely categorical criterion. This is true: additive functors are precisely those that preserve finite products.

Proposition 2.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ is additive if and only if it commutes with finite products.*

Proof sketch. For a full proof, see [KS06, pp. 173–174, Prop. 8.2.15]. This follows essentially from the fact that, for two functors $F, F'\colon \mathcal{C} \to \mathbf{Ab}$ commuting with finite products, the obvious morphism

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Ab})}(F, F') \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(U \circ F, U \circ F'), \quad \eta \mapsto U\eta$$

given by composing with the forgetful functor $U\colon \mathbf{Ab} \to \mathbf{Set}$ is an isomorphism (see [KS06, p. 173, Prop. 8.2.12]). The argument is then:

Fix some $A \in \mathcal{C}$. We then have two functors $\alpha, \beta\colon \mathcal{C} \to \mathbf{Ab}$ given by

$$\alpha\colon Y \mapsto \mathrm{Hom}_{\mathcal{C}}(A, Y), \quad \beta\colon Y \mapsto \mathrm{Hom}_{\mathcal{D}}(F(A), F(Y)).$$

Then α commutes with finite products, and if F commutes with finite products, then β also does. In that case, the functors $U \circ \alpha$ and $U \circ \beta$ also commute with products, and so the canonical natural transformation $U \circ \alpha \to U \circ \beta$, $f \mapsto F(f)$, lifts to a natural transformation $\alpha \to \beta$. In light of what this lifting is, this says any set map $\mathrm{Hom}_{\mathcal{C}}(A, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(Y))$ is also a group homomorphism.

Conversely, suppose F is additive. For any $A, B \in \mathcal{C}$, we have a characterization of the product $A \times B$ from Proposition 2.2. By the additivity of F, the same relations hold for $F(A \times B)$. ☐

Using the characterization of additive categories from Remark 2.9, we have the following proposition, which in some cases lets us deduce that a category is additive.

Proposition 2.12. *Let $\mathcal{C}$ be an additive category, and let $\mathcal{D}$ be a category where finite products and coproducts exist and coincide. Suppose there exists a functor $F\colon \mathcal{C} \to \mathcal{D}$ which is essentially surjective and preserves finite products and finite coproducts. Then this induces a unique additive structure on $\mathcal{D}$ such that F is additive.*

Proof sketch. We just have to check the conditions of Remark 2.9. By assumption, (a)–(c) are verified in $\mathcal{D}$, and so we just need to check (d). Let $X \in \mathcal{D}$. Since F is essentially surjective, we know that $X \cong F(A)$ for some $A \in \mathcal{C}$. Since $\mathcal{C}$ is additive, we have a morphism $a\colon A \to A$ such that

$$A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{(a,\mathrm{id}_A)} A \oplus A \xrightarrow{\nabla_A} A$$

is zero. Applying F and using that it preserves both finite products and coproducts, we see that

$$F(A) \xrightarrow{\Delta_{F(A)}} F(A) \oplus F(A) \xrightarrow{(F(a),\mathrm{id}_{F(A)})} F(A) \oplus F(A) \xrightarrow{\nabla_{F(A)}} F(A)$$

is zero, so $F(a)$ verifies (d). Therefore, $\mathcal{D}$ is additive. That F is additive follows since it preserves finite products by Proposition 2.11. ☐

2.2 ABELIAN CATEGORIES

While in additive categories we have access to addition of morphisms, this is not enough structure to do homological algebra. The category **Ab** of Abelian groups satisfies more properties. For example, we may form kernels, cokernels, and images, and we have various isomorphism theorems. A natural setting to generalize this is in the framework of Abelian categories.

Definition 2.13. Let $\mathcal{C}$ be a category with a zero object, and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. The *kernel* of f is the pullback

$$\begin{tikzcd}
\ker f \arrow[r, dashed] \arrow[d, dashed] & 0 \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and the *cokernel* of f is the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, dashed] \\
0 \arrow[r, dashed] & \operatorname{coker} f.
\end{tikzcd}$$

The *image* of f is

$$\operatorname{im} f := \ker(Y \to \operatorname{coker} f),$$

while the *coimage* of f is

$$\operatorname{coim} f := \operatorname{coker}(\ker f \to X).$$

Remark 2.14. Note that the map $\ker f \to X$ is automatically a monomorphism. Similarly, the map $Y \to \operatorname{coker} f$ is automatically an epimorphism.

Remark 2.15. Thus, the image is defined by the universal property that any morphism $Z \to Y$ such that $Z \to Y \to \operatorname{coker} f$ composes to zero factorizes uniquely through the map $\operatorname{im} f \to Y$. Thus, since $X \xrightarrow{f} Y \to \operatorname{coker} f$ composes to zero, there is a canonical map $X \to \operatorname{im} f$.

Remark 2.16. Similarly, the coimage is defined by the universal property that any morphism $X \to Z$ for which the composition $\ker f \to X \to Z$ is zero factorizes uniquely through the morphism $X \to \operatorname{coim} f$. Thus, since $\ker f \to X \to Y$ composes to zero, there is a canonical map $\operatorname{coim} f \to Y$.

Remark 2.17. Furthermore, one observes that $\operatorname{coim} f \to Y \to \operatorname{coker} f$ composes to zero: composing $f : X \to Y$ with the map $Y \to \operatorname{coker} f$ gives the zero map, and therefore by definition the composition $X \twoheadrightarrow \operatorname{coim} f \to Y \twoheadrightarrow \operatorname{coker} f$ is zero, and thus since $X \twoheadrightarrow \operatorname{coim} f$ is epic we obtain the result. Using this fact, we may apply the universal property of the image to produce a canonical map $\operatorname{coim} f \to \operatorname{im} f$. The situation is summarized in the following diagram:

$$\begin{tikzcd}
\ker f \arrow[r, hook] \arrow[dr, "0"'] & X \arrow[r, "f"] \arrow[d, two heads] \arrow[dr] & Y \arrow[r, two heads] & \operatorname{coker} f. \\
& \operatorname{coim} f \arrow[r, dashed] \arrow[ur] & \operatorname{im} f \arrow[u, hook] \arrow[ur, "0"'] &
\end{tikzcd}$$

Remark 2.18. If we use some abusive notation, then for a morphism $g : A \to B$ we can write $\operatorname{coker} g =: B/g(A)$. If g is a monomorphism, we can then abuse the notation even more and write $\operatorname{coker} g = B/A$. This notation allows us to observe that $\operatorname{im} f = \ker(Y \to Y/f(X))$. Similarly, $\operatorname{coim} f = X/\ker f$.

Definition 2.19. An *Abelian category* is an additive category $\mathcal{C}$ such that

(a) for every morphism f in $\mathcal{C}$, the kernel $\ker f$ and cokernel $\operatorname{coker} f$ exist, and

(b) for every morphism f in $\mathcal{C}$, the canonical map $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism.

Remark 2.20. Using the notation from the recent remark, we then see that (b) in the above definition essentially corresponds to a version of the first isomorphism theorem. In particular, it asserts that $X/\ker f \cong \operatorname{im} f$.

Remark 2.21. An immediate consequence of the definition is that any morphism factors as an epimorphism followed by a monomorphism. In particular, consider a morphism $f : X \to Y$. Then, by identifying $\operatorname{im} f$ and $\operatorname{coim} f$, we obtain the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads] & & Y \\
& \operatorname{im} f \arrow[ur, hook] &
\end{tikzcd}$$

which gives the desired factorization. Note that the identification of $\operatorname{im} f$ and $\operatorname{coim} f$ is important: the first gives the monomorphism, and the second gives the epimorphism.

Using Abelian categories, we can now do some homological algebra. Actually, the following definition works just as well in an additive category, so that's where we will state it.

Definition 2.22. Let $\mathcal{C}$ be an additive category. A *chain complex* in $\mathcal{C}$ is a sequence of objects and morphisms

$$\cdots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots$$

in $\mathcal{C}$ such that for all $i \in \mathbb{Z}$, we have $d^{i+1} \circ d^i = 0$.

Remark 2.23. We can also formulate the definition in a more categorical way as follows: a *graded* $\mathcal{C}$-object will be a functor $C^\bullet\colon \mathbb{Z} \to \mathcal{C}$, $i \mapsto C^i$, where we regard $\mathbb{Z}$ as a discrete category, i.e. with only identity morphisms. This assembles into a category, namely the functor category $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$. Now let $C^\bullet \in \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$. We can define, for any j, the shift $C(j)^\bullet$ by $C(j)^i = C^{i+j}$, and this clearly extends to an automorphism $\mathrm{Fun}(\mathbb{Z}, \mathcal{C}) \to \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$. A chain complex is now a graded $\mathcal{C}$-object $C^\bullet$ together with a morphism $d_C : C^\bullet \to C(1)^\bullet$ such that $d_C(1) \circ d_C = 0$. Note that this notion is the same as a *differential graded* $\mathcal{C}$-object. This justifies the following notational convenience: the pair $(C^\bullet, d)$ is a chain complex if $d^2 = 0$.

Chain complexes in Abelian categories have *cohomology.* In particular, consider a chain complex

$$\cdots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots$$

in an Abelian category $\mathcal{A}$. Then we note that the requirement that $d^2 = 0$ implies that d^{i-1} factors through the kernel of d^i. In fact, since $C^{i-1} \twoheadrightarrow \operatorname{im} d^{i-1}$ is an epimorphism, this shows that the composition $\operatorname{im} d^{i-1} \hookrightarrow C^i \to C^{i+1}$ is zero. Similarly, since the map $\operatorname{im} d^i \hookrightarrow C^{i+1}$ is a monomorphism, the composition $C^{i-1} \to C^i \twoheadrightarrow \operatorname{im} d^i$ is zero. In conclusion, we have the commutative diagram (taken directly from [KS06])

$$\begin{tikzcd}
& \operatorname{im} d^{i-1} \arrow[rr, dashed, "\phi"] \arrow[dr, hook] & & \ker d^i \arrow[dl, hook'] \arrow[dr, "0"] & \\
C^{i-1} \arrow[ur, two heads] \arrow[rr, "d^{i-1}"] \arrow[dr, "0"'] & & C^i \arrow[rr, "d^i"] \arrow[dl, two heads] \arrow[dr, two heads] & & C^{i+1} \\
& \operatorname{coker} d^{i-1} \arrow[rr, dashed, "\psi"] & & \operatorname{im} d^i \arrow[ur, hook] &
\end{tikzcd}$$

Proposition 2.24. *The morphism ϕ (resp. ψ) in the above diagram is a monomorphism (resp. an epimorphism).*

Proof. Consider a map $z\colon Z \to \operatorname{im} d^{i-1}$ such that $\phi \circ z = 0$. Composing with the monomorphism $\ker d^i \hookrightarrow C^i$ and using commutativity shows that $\phi \circ z = 0$ if and only if the composition $Z \xrightarrow{z} \operatorname{im} d^{i-1} \hookrightarrow C^i$ is zero, which happens if and only if $z = 0$. The proof for ψ is dual. ■

Thus, we actually have a diagram

$$\begin{tikzcd}
& \operatorname{im} d^{i-1} \arrow[rr, dashed, hook, "\phi"] \arrow[dr, hook] & & \ker d^i \arrow[dl, hook'] \arrow[dr, "0"] & \\
C^{i-1} \arrow[ur, two heads] \arrow[rr, "d^{i-1}"] \arrow[dr, "0"'] & & C^i \arrow[rr, "d^i"] \arrow[dl, two heads] \arrow[dr, two heads] & & C^{i+1}. \\
& \operatorname{coker} d^{i-1} \arrow[rr, dashed, two heads, "\psi"] & & \operatorname{im} d^i \arrow[ur, hook] &
\end{tikzcd}$$

Let u^i be the composition of the morphisms $\ker d^i \hookrightarrow C^i \twoheadrightarrow \operatorname{coker} d^{i-1}$. Then ϕ factors through the kernel of u^i since $u^i \circ \phi = 0$ (which can be checked by using that various morphisms are mono/epimorphisms), and the induced map $\operatorname{im} d^{i-1} \to \ker u^i$ is a monomorphism (checked similarly as before). Furthermore, there is a monomorphism $\ker u^i \hookrightarrow \operatorname{im} d^{i-1}$ essentially by definition of the image. Thus we have an expanded diagram of the form

$$\begin{tikzcd}
& & \ker u^i \arrow[dl, dashed, hook, shift right] \arrow[dr, hook] & & \\
& \operatorname{im} d^{i-1} \arrow[ur, dashed, hook, shift right] \arrow[rr, hook, "\phi"] \arrow[dr, hook] & & \ker d^i \arrow[dl, hook] \arrow[dr, "0"] & \\
C^{i-1} \arrow[ur, two heads] \arrow[rr, "d^{i-1}"] \arrow[dr, "0"'] & & C^i \arrow[rr, "d^i"] \arrow[dl, two heads] \arrow[dr, two heads] & & C^{i+1} \\
& \operatorname{coker} d^{i-1} \arrow[rr, two heads, "\psi"] & & \operatorname{im} d^i \arrow[ur, hook] &
\end{tikzcd}$$

where the dashed arrows are unique in making this diagram commute. It is now easily seen that $\ker u^i$ satisfies the universal property of $\operatorname{im} d^{i-1}$ by using the monomorphisms $\operatorname{im} d^{i-1} \hookrightarrow \ker u^i$ and $\ker u^i \hookrightarrow \operatorname{im} d^{i-1}$. Therefore, these monomorphisms are actually isomorphisms. The same reasoning can be run in dual to finally obtain the following rather large commutative diagram, which summarizes the situation:

$$\begin{tikzcd}
& & \ker u^i \arrow[dl, "\sim"'] \arrow[dr, hook] & & \\
& \operatorname{im} d^{i-1} \arrow[rr, hook] \arrow[dr, hook] & & \ker d^i \arrow[dl, hook] \arrow[dr, "0"] & \\
C^{i-1} \arrow[ur, two heads] \arrow[rr, "d^{i-1}"] \arrow[dr, "0"'] & & C^i \arrow[rr, "d^i"] \arrow[dl, two heads] \arrow[dr, two heads] & & C^{i+1} \\
& \operatorname{coker} d^{i-1} \arrow[rr, two heads] \arrow[dr, two heads] & & \operatorname{im} d^i \arrow[ur, hook] \arrow[dl, "\sim"] & \\
& & \operatorname{coker} u^i & &
\end{tikzcd}$$

As a result, using the fact that $\operatorname{im} u^i \cong \operatorname{coim} u^i$, we have the following natural isomorphisms (taken almost verbatim from [KS06]):

$$\begin{aligned} \operatorname{im} u^i &\cong \ker(\operatorname{coker} d^{i-1} \twoheadrightarrow \operatorname{im} d^i) \cong \ker(\operatorname{coker} d^{i-1} \to C^{i+1}) \\ &\cong \operatorname{coker}(\operatorname{im} d^{i-1} \hookrightarrow \ker d^i) \cong \operatorname{coker}(C^{i-1} \to \ker d^i). \end{aligned}$$

Two of these isomorphisms follow by checking universal properties (and using that the appropriate maps are mono/epimorphisms). Finally, we can define cohomology.

Definition 2.25. Let $(C^\bullet, d)$ be a chain complex in an Abelian category $\mathcal{A}$. The *cohomology* of $C^\bullet$ at $i \in \mathbb{Z}$ is

$$\mathrm{H}^i(C^\bullet) := \operatorname{coker}(\operatorname{im} d^{i-1} \hookrightarrow \ker d^i),$$

or any of the other equivalent choices from above.

Remark 2.26. Notice immediately that in the case where $\mathcal{A} = \mathbf{Ab}$, we have $\mathrm{H}^i(C^\bullet) = \ker d^i / \operatorname{im} d^{i-1}$.

Note that using the isomorphisms we have above, we see that

$$\begin{aligned} u^i = 0 \iff \operatorname{coker} d^{i-1} \xrightarrow{\sim} \operatorname{im} d^i \iff \operatorname{coker} d^{i-1} \hookrightarrow C^{i+1} \\ \iff \operatorname{im} d^{i-1} \xrightarrow{\sim} \ker d^i \iff C^{i-1} \twoheadrightarrow \ker d^i \iff \mathrm{H}^i(C^\bullet) = 0. \end{aligned}$$

Definition 2.27. Let $(C^\bullet, d)$ be a chain complex in an Abelian category $\mathcal{A}$. We say $C^\bullet$ is *exact* at $i \in \mathbb{Z}$ if $\mathrm{H}^i(C^\bullet) = 0$, i.e. if any of the above equivalent conditions are satisfied. A chain complex is a *long exact sequence* (or just an *exact sequence*) if it is exact at every $i \in \mathbb{Z}$. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

Using the language of Abelian categories, we can prove the following version of the first isomorphism theorem regarding short exact sequences:

Theorem 2.28 (First isomorphism theorem)**.** *Let $\mathcal{A}$ be an Abelian category, and suppose we have a short exact sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{A}$. Then

$$\mathrm{im}(X \to Y) \cong X \quad \textit{and} \quad \mathrm{coker}(X \to Y) \cong Z.$$

Proof. To see the first claim, note that

$$\mathrm{im}(X \to Y) = \ker(Y \to \mathrm{coker}(X \to Y)) \cong \mathrm{coker}(\ker(X \to Y) \to X) \cong \mathrm{coker}(0 \to X) = X.$$

To see the second claim, note that we have

$$\begin{aligned} Z \cong \mathrm{im}(Y \to Z) &= \ker(Z \to \mathrm{coker}(Y \to Z)) \\ &\cong \mathrm{coker}(\ker(Y \to Z) \to Y) \\ &\cong \mathrm{coker}(\mathrm{im}(X \to Y) \to Y) \cong \mathrm{coker}(X \to Y) \end{aligned}$$

which completes the proof. ■

It will be important later to discuss issues of exactness in (hearts of) triangulated categories (with t-structures). Thus, we include here the definition of an *exact* functor:

Definition 2.29. Let $\mathcal{A}, \mathcal{B}$ be Abelian categories. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is *left* (resp. *right*) *exact* if it sends an exact sequence $0 \to X \to Y \to Z \to 0$ to an exact sequence

$$0 \to F(X) \to F(Y) \to F(Z) \quad (\text{resp. } F(X) \to F(Y) \to F(Z) \to 0).$$

We say a functor is *exact* if it is left and right exact.

We have defined chain complexes in an Abelian category $\mathcal{A}$ essentially as objects $C^\bullet$ of the category $\mathrm{Fun}(\mathbb{Z}, \mathcal{A})$ together with a natural transformation $d_C : C^\bullet \to C(1)^\bullet$. This makes it quite easy to see that we can assemble this data into a category:

Definition 2.30. Let $\mathcal{A}$ be an Abelian category. The *category of chain complexes* $\mathbf{C}(\mathcal{A})$ in $\mathcal{A}$ is the category in which the objects are chain complexes $(C^\bullet, d_C)$ and the morphisms are morphisms of graded objects (i.e. natural transformations) $\eta\colon A^\bullet \to B^\bullet$ such that $d_B \circ \eta = \eta(1) \circ d_A$, where $\eta(1)$ denotes the induced natural transformation $A(1)^\bullet \to B(1)^\bullet$.

Remark 2.31. Thus a morphism $A^\bullet \to B^\bullet$ is a collection of morphisms $A^i \to B^i$ fitting into the diagram

$$\begin{tikzcd} \cdots \arrow[r] & A^{i-1} \arrow[r, "d"] \arrow[d] & A^i \arrow[r, "d"] \arrow[d] & A^{i+1} \arrow[r, "d"] \arrow[d] & \cdots \\ \cdots \arrow[r] & B^{i-1} \arrow[r, "d"] & B^i \arrow[r, "d"] & B^{i+1} \arrow[r, "d"] & \cdots \end{tikzcd}$$

where we omit the decorations for convenience.

Proposition 2.32. *The category* $\mathbf{C}(\mathcal{A})$ *is an Abelian category.*

Proof sketch. Essentially, this just depends on the fact that for any category $\mathcal{C}$, the category $\mathrm{Fun}(\mathcal{C},\mathcal{A})$ is Abelian. This itself basically relies on the fact that you can essentially just check the axioms "point-wise." Checking that all this is compatible with the differentials on the chain complexes is not so hard. ☐

Let $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ be chain complexes in an Abelian category $\mathcal{A}$, and suppose we have a morphism $f\colon A^\bullet \to B^\bullet$. This will induce a morphism $\mathrm{H}^i(A^\bullet) \to \mathrm{H}^i(B^\bullet)$ for each $i \in \mathbb{Z}$. To see this, first observe that since $f^{i+1} \circ d_A^i = d_B^i \circ f^i$, the composition $\ker d_A^i \hookrightarrow A^i \xrightarrow{f^i} B^i$ factors through $\ker d_B^i$. We then note that to get a morphism $\mathrm{H}^i(A^\bullet) \to \mathrm{H}^i(B^\bullet)$, it suffices to show that the composition of the canonical morphisms

$$\operatorname{im} d_A^{i-1} \hookrightarrow \ker d_A^i \to \ker d_B^i \twoheadrightarrow \mathrm{H}^i(B^\bullet)$$

is zero. To do this, we produce a canonical morphism $\operatorname{im} d_A^{i-1} \to \operatorname{im} d_B^{i-1}$. The following diagram displays both steps:

$$\begin{tikzcd}
A^{i-1} \arrow[r, two heads] \arrow[drr, "d_A^{i-1}"'] \arrow[dd, "f^{i-1}"'] & \operatorname{im} d_A^{i-1} \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, dashed] & & \ker d_A^i \arrow[r, two heads] \arrow[dl, hook'] \arrow[dd] & \mathrm{H}^i(A^\bullet) \arrow[dd, dashed, "\mathrm{H}^i(f)"] \\
& & A^i \arrow[dd, "f^i"] & & \\
B^{i-1} \arrow[r, two heads] \arrow[drr, "d_B^{i-1}"'] & \operatorname{im} d_B^{i-1} \arrow[rr, hook] \arrow[dr, hook] & & \ker d_B^i \arrow[r, two heads] \arrow[dl, hook'] & \mathrm{H}^i(B^\bullet) \\
& & B^i \arrow[rr, two heads] & & \operatorname{coker} d_B^{i-1}.
\end{tikzcd}$$

In particular, to produce the map between the images, it suffices to show that the composition

$$\operatorname{im} d_A^{i-1} \hookrightarrow \ker d_A^i \to \ker d_B^i \hookrightarrow B^i \twoheadrightarrow \operatorname{coker} d_B^{i-1}$$

is zero. However, this follows by the commutativity of the diagram together with the fact that the map $A^{i-1} \twoheadrightarrow \operatorname{im} d_A^{i-1}$ is an epimorphism. The morphism $\mathrm{H}^i(f) : \mathrm{H}^i(A^\bullet) \to \mathrm{H}^i(B^\bullet)$ then follows by commutativity, since the composition

$$\operatorname{im} d_A^{i-1} \to \operatorname{im} d_B^{i-1} \hookrightarrow \ker d_B^i \twoheadrightarrow \mathrm{H}^i(B^\bullet)$$

is zero. Thus we have essentially proven

Proposition 2.33. *Let* $\mathcal{A}$ *be an Abelian category, and let* $f\colon A^\bullet \to B^\bullet$ *be a morphism of chain complexes in* $\mathcal{A}$*. Then this induces a canonical morphism*

$$\mathrm{H}^i(f)\colon \mathrm{H}^i(A^\bullet) \to \mathrm{H}^i(B^\bullet)$$

for every $i \in \mathbb{Z}$*. This data assembles into an additive functor* $\mathrm{H}^i\colon \mathbf{C}(\mathcal{A}) \to \mathcal{A}$*.*

Proof sketch. We have provided most of the data, so we know what H^i does on objects and morphisms. What remains to check is that it preserves composition, but this can be checked by using that every morphism was produced by a universal property (and thus are unique in making all the diagrams commute). The composition "on the nose" would also make the diagrams commute, and so we must have $\mathrm{H}^i(g \circ f) = \mathrm{H}^i(g) \circ \mathrm{H}^i(f)$ for all $f : A^\bullet \to B^\bullet$, $g : B^\bullet \to C^\bullet$. Essentially the same reasoning shows that $\mathrm{H}^i(f + g) = \mathrm{H}^i(f) + \mathrm{H}^i(g)$ for all morphisms $f, g \colon A^\bullet \to B^\bullet$. □

Finally, we will make use of the famous *five lemma* in several places throughout the text. The formulation we take here is from [KS06].

Lemma 2.34. [KS06, p. 181, Lemma 8.3.13]. *Let $\mathcal{A}$ be an Abelian category, and suppose we have a commutative diagram*

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d, "f^0"] & X^1 \arrow[r] \arrow[d, "f^1"] & X^2 \arrow[r] \arrow[d, "f^2"] & X^3 \arrow[d, "f^3"] \\
Y^0 \arrow[r] & Y^1 \arrow[r] & Y^2 \arrow[r] & Y^3
\end{tikzcd}$$

in $\mathcal{A}$ whose rows form a complex, and assume that $X^1 \to X^2 \to X^3$ and $Y^0 \to Y^1 \to Y^2$ are exact sequences.

(i) If f^0 is an epimorphism and f^1, f^3 are monomorphisms, then f^2 is a monomorphism.

(ii) If f^3 is a monomorphism and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.

Corollary 2.35 (Classical five lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & X^2 \arrow[r] \arrow[d, dashed] & X^3 \arrow[r] \arrow[d] & X^4 \arrow[d] \\
Y^0 \arrow[r] & Y^1 \arrow[r] & Y^2 \arrow[r] & Y^3 \arrow[r] & Y^4
\end{tikzcd}$$

with exact rows. If all vertical arrows aside from the dashed one are isomorphisms, then so is the dashed one.

Proof. Applying (i) in Lemma 2.34, one sees the dashed arrow is a monomorphism. Applying (ii), one sees it is an epimorphism. Therefore, since isomorphisms in Abelian categories are exactly those which are monic and epic, this implies the dashed arrow is an isomorphism. ■

3 Triangulated Categories

Triangulated categories were introduced by Grothendieck and developed by Verdier in his 1967 thesis [Ver96]. They aim to get around the fact that derived categories and homotopy categories of complexes fail to be Abelian by recognizing that these still maintain additivity along with knowledge about exact sequences. Triangulated categories provide a general theory of additive categories wherein one has "short exact sequences" which give rise to "long exact sequences" in an appropriate sense, and furthermore allow one to have an appropriate non-trivial ambient category for derived categories to live inside as objects (in particular, we have a notion of triangulated functors; see Definition 3.9).

To a large extent, this section follows [KS06] although we choose slightly different conventions. Some results about the Verdier quotient come from [Nee01]. The proof that the localization of an additive category (by a suitable class of morphisms) is additive can be found in, for example, [Kra22, p. 29, Lemma 2.2.1].

3.1 (Pre-)Triangulated Categories

Definition 3.1. Let $\mathcal{D}$ be a category, and let $(-)[1]\colon \mathcal{D} \to \mathcal{D}$ be an automorphism. A *triangle* (X, Y, Z, u, v, w) with respect to $(-)[1]$ is a sequence of objects $X, Y, Z \in \mathcal{D}$ and morphisms $u\colon X \to Y$, $v\colon Y \to Z$, and $w\colon Z \to X[1]$, i.e. a diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1]. \end{tikzcd}$$

A *morphism of triangles* $(X, Y, Z, u, v, w) \to (X', Y', W', u', v', w')$ is a triple (f, g, h) of morphisms $f\colon X \to X'$, $g\colon Y \to Y'$, and $h\colon Z \to Z'$ fitting into a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "{f[1]}"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

We say a triangle (X, Y, Z, u, v, w) is a *candidate triangle* if $w \circ v = 0$ and $v \circ u = 0$.

Definition 3.2. A *pre-triangulated category* is an additive category $\mathcal{D}$ together with an additive automorphism $(-)[1]\colon \mathcal{D} \to \mathcal{D}$, where we write $(-)[n]$, $n \in \mathbb{Z}$, for $(-)[1]$ applied n times (with $(-)[-1]$ being the inverse of $(-)[1]$) and a class of triangles with respect to $(-)[1]$ stable under isomorphism, called *distinguished triangles*, satisfying

(TR1) for any $X \in \mathcal{D}$, the triangle

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] & X \arrow[r] & 0 \arrow[r] & X[1] \end{tikzcd}$$

is a distinguished triangle, and for any morphism $f\colon X \to Y$ in $\mathcal{D}$, there exists some $Z \in \mathcal{D}$ (called a *cone* of f) and a distinguished triangle

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r] & Z \arrow[r] & X[1], \end{tikzcd}$$

(TR2) given the two triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

and

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

if one is distinguished, then so is the other, and

(TR3) for any commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

where the rows are distinguished triangles, there exists some not necessarily unique

$$h \colon Z \to Z'$$

such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

commutes.

Remark 3.3. We will sometimes, although not necessarily consistently, denote a choice of Z in (TR1) by C_f and refer to it as "the" cone. Note that the choice here is not canonical at all (although all choices are *non-canonically* isomorphic, see Proposition 3.17), so this is largely useful for remembering where the object came from. In these instances, we will sometimes write K_f for $C_f[-1]$ and call it a *cocone.*

Remark 3.4. Distinguished triangles are essentially supposed to play the role of (short) exact sequences in Abelian categories, with the caveat that we no longer have access to kernels and cokernels with which to actually formulate exactness. This, however, does suggest the intuition that in a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

one should think of X as being a kind of "weak kernel" of the map $Y \to Z$, and of Z as being a kind of "weak cokernel" of the map $X \to Y$. In fact, this intuition can be made formal in several ways: using cohomological functors, which we do in Proposition 3.20 using Proposition 3.15, and later, using t-structures, we will see (in Theorem 5.20) that in select situations these "weak (co)kernels" are actual (co)kernels.

Definition 3.5. One says a pre-triangulated category $\mathcal{D}$ is *triangulated* if it further satisfies

(TR4) if there are three distinguished triangles

$$\begin{cases} X \xrightarrow{u} Y \longrightarrow Z \longrightarrow X[1] \\ X \xrightarrow{v\circ u} Y' \longrightarrow Z' \longrightarrow X[1] \\ Y \xrightarrow{v} Y' \longrightarrow Y'' \longrightarrow Y[1] \end{cases}$$

then there exists $f\colon Z \to Z'$, $g\colon Z' \to Y''$, and $h\colon Y'' \to Z[1]$ such that

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed, "f"] & X[1] \arrow[d, equal] \\
X \arrow[r, "v\circ u"] \arrow[d, "u"] & Y' \arrow[r] \arrow[d, equal] & Z' \arrow[r] \arrow[d, dashed, "g"] & X[1] \arrow[d, "u[1]"] \\
Y \arrow[r, "v"] \arrow[d] & Y' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d, equal] & Y[1] \arrow[d] \\
Z \arrow[r, dashed, "f"] & Z' \arrow[r, dashed, "g"] & Y'' \arrow[r, dashed, "h"] & Z[1]
\end{tikzcd}$$

commutes, and the bottom row

$$\begin{tikzcd}
Z \arrow[r, dashed, "f"] & Z' \arrow[r, dashed, "g"] & Y'' \arrow[r, dashed, "h"] & Z[1]
\end{tikzcd}$$

is a distinguished triangle. This axiom is sometimes called the *octahedral axiom,* since it is possible to assemble the above data into a diagram shaped like an octahedron.

Remark 3.6. We can, and frequently will, write the diagram in (TR4) as

$$\begin{tikzcd}
& & & {} & & \\
& & Z \arrow[ur] \arrow[dr, dashed, "f"] & & {} & \\
& Y \arrow[ur] \arrow[dr] & & Z' \arrow[ur] \arrow[ddr, dashed, "g"] & & \\
& & Y' \arrow[ur] \arrow[drr] & & & \\
X \arrow[uur] \arrow[urr] & & & & Y'' \arrow[dr] \arrow[dr, dashed, "h"'] & \\
& & & & & {}
\end{tikzcd}$$

where we note that we omit the shift operations.

Remark 3.7. We can think of (TR4) as giving us a prefered choice of morphism h in (TR3) in certain situations to make up for the fact that taking cones is not functorial. In particular, suppose we have a map $u : X \to Y$ and a map $v : Y \to Y'$. We then complete these to distinguished triangles $X \to Y \to C_u$ and $Y \to Y' \to C_v$ and note that we have the following diagram of solid arrows

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & C_u \arrow[r] \arrow[d, dashed, "h"] & X[1] \arrow[d, "u[1]"] \\
Y \arrow[r, "v"] & Y' \arrow[r] & C_v \arrow[r] & Y[1]
\end{tikzcd}$$

which then allows us, via (TR3), to obtain the dashed arrow $h\colon C_u \to C_v$ giving us a morphism of distinguished triangles. However, since we lack functoriality, we have no information on how this relates to the cone of $v \circ u$, thereby leaving us with very few compatibility relations. The octahedral axiom (TR4) ameliorates this by allowing us to take h in (TR3) as the composition $g \circ f$ of morphisms $f\colon C_u \to C_{v \circ u}$ and $g\colon C_{v\circ u} \to C_v$ sitting in the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "v"] & C_u \arrow[r] \arrow[d, dashed, "f"] & X[1] \arrow[d, equal] \\
X \arrow[r, "v \circ u"] \arrow[d, "u"] & Y' \arrow[r] \arrow[d, equal] & C_{v \circ u} \arrow[r] \arrow[d, dashed, "g"] & X[1] \arrow[d, "{u[1]}"] \\
Y \arrow[r, "v"] & Y' \arrow[r] & C_v \arrow[r] \arrow[d, dashed] & Y[1] \arrow[dl] \\
 & & C_u[1] &
\end{tikzcd}$$

thereby giving us a relation between the cones of u, v and $v \circ u$.

Remark 3.8. The interpretation of cones as "weak cokernels" suggests another intuition for (TR4). Consider the situation in (TR4), and informally write $Z = Y/X$, $Z' = Y'/X$, and $Y'' = Y'/Y$, as indicated by pretending these are all short exact sequences. Then (TR4) gives us a "short exact sequence" identifying the fact that

$$(Y/X)/(Y'/X) = Y/Y'.$$

That is, (TR4) can be thought of as saying that the third isomorphism theorem is true. Compare this with the fact that one of the axioms of Abelian categories is that the *first* isomorphism theorem is true.

Definition 3.9. Let $\mathcal{D}$ and $\mathcal{D}'$ be two triangulated categories. A *triangulated functor* is an additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ such that there is a natural isomorphism $F \circ [1]_{\mathcal{D}} \cong [1]_{\mathcal{D}'} \circ F$, and for all distinguished triangles

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{D}$, the triangle

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(X)[1]$$

obtained by applying F and using that $F(X[1]) \cong F(X)[1]$ is a distinguished triangle in $\mathcal{D}'$.

Definition 3.10. Let $\mathcal{D}$ be a triangulated category. A subcategory $\mathcal{C}$ in $\mathcal{D}$ together with a triangulated structure is a *triangulated subcategory* if the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ is a triangulated functor.

Proposition 3.11. *Let $\mathcal{D}$ and $\mathcal{E}$ be two triangulated categories, and let $F : \mathcal{D} \to \mathcal{E}$ be a triangulated functor. If $G : \mathcal{E} \to \mathcal{D}$ is a left or right adjoint to F, then G is also a triangulated functor.*

Proof sketch. For a full proof, see [Nee01, Lemma 5.3.6]. We sketch this in the case where G is right adjoint to F. Let $X \in \mathcal{E}$, $Y \in \mathcal{D}$. Then we have isomorphisms, natural in X and Y,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(Y, G(X[1])) &\cong \mathrm{Hom}_{\mathcal{E}}(F(Y), X[1]) \cong \mathrm{Hom}_{\mathcal{E}}(F(Y)[-1], X) \\ &\cong \mathrm{Hom}_{\mathcal{E}}(F(Y[-1]), X) \cong \mathrm{Hom}_{\mathcal{D}}(Y[-1], G(X)) \cong \mathrm{Hom}_{\mathcal{D}}(Y, G(X)[1]).\end{aligned}$$

By the Yoneda lemma, this gives a natural isomorphism $G \circ [1] \cong [1] \circ G$. We prove that G preserves distinguished triangles by using the five lemma (Corollary 2.35). In particular, consider a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{E}$. Applying G to the morphism $X \to Y$ and taking the cone, we obtain a distinguished triangle

$$G(X) \longrightarrow G(Y) \longrightarrow C \longrightarrow G(X)[1].$$

Applying F gives a distinguished triangle

$$F(G(X)) \longrightarrow F(G(Y)) \longrightarrow F(C) \longrightarrow F(G(X))[1]$$

in $\mathcal{E}$ and one now makes use of the counit $F \circ G \to \mathrm{id}$ and (TR3) to produce a morphism to the original distinguished triangle. The new map $F(C) \to Z$ then defines a map $\mathrm{Hom}_{\mathcal{D}}(W, C) \to \mathrm{Hom}_{\mathcal{E}}(F(W), Z)$ which is an isomorphism by the five lemma. □

Having a pre-triangulated structure allows us to, in an elementary way, define the notion of *cohomological functors*. In particular, a distinguishing property of cohomology is that it turns *short* exact sequences into *long* exact sequences. The structure on a pre-triangulated category consists exactly of information that allows us to formalize this. The distinguished triangles play the role of the short exact sequences, and the shift allows us to extend this to long sequences.

Definition 3.12. Let $\mathcal{D}$ be a pre-triangulated category, and let $\mathcal{A}$ be an Abelian category. An additive functor $F \colon \mathcal{D} \to \mathcal{A}$ is *cohomological* if for any distinguished triangle $X \to Y \to Z \to X[1]$, the sequence

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z)$$

in $\mathcal{A}$ is exact.

Remark 3.13. Here is an explicit way to see that this definition makes sense. Consider a distinguished triangle as above. We may extend it to a sequence of morphisms

$$\cdots \longrightarrow Y[-1] \longrightarrow Z[-1] \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] \longrightarrow Y[1] \longrightarrow \cdots$$

which after applying F gives a sequence

$$\cdots \longrightarrow F(Y[-1]) \longrightarrow F(Z[-1]) \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(X[1]) \longrightarrow F(Y[1]) \longrightarrow \cdots$$

and the requirement that F is cohomological is exactly that this sequence of morphisms is a long exact sequence (at least after potentially switching signs in some places). If we temporarily write $F^n := F \circ [n]$, then this has the form

$$\cdots \longrightarrow F^{-1}(Y) \longrightarrow F^{-1}(Z) \longrightarrow F^0(X) \longrightarrow F^0(Y) \longrightarrow F^0(Z) \longrightarrow F^1(X) \longrightarrow F^1(Y) \longrightarrow \cdots$$

which is suggestive when compared with the classical long exact sequence in cohomology from topology or geometry.

If cohomological functors are to be reasonable, then there should exist reasonable examples. The following proposition is an important result which says that all objects of $\mathcal{D}$ induce a cohomological functor via the Yoneda embedding. One can think of it as a kind of “triangulated Yoneda lemma.”

Lemma 3.14. *Let $\mathcal{D}$ be a pre-triangulated category, and consider a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1].$$

Then $g \circ f = 0$.

Proof. By (TR1) and (TR3), we have a morphism of triangles

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "\mathrm{id}"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

which, by commutativity, gives that $g \circ f = 0$. ■

Proposition 3.15. *Let $\mathcal{D}$ be a pre-triangulated category. Then, for any $X \in \mathcal{D}$, the functors*

$$\mathrm{Hom}_{\mathcal{D}}(X, -)\colon \mathcal{D} \to \mathbf{Ab} \quad \textit{and} \quad \mathrm{Hom}_{\mathcal{D}}(-, X)\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Ab}$$

are cohomological.

Proof. Fix a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

and an object $E \in \mathcal{D}$. We need to show that

$$\mathrm{Hom}_{\mathcal{D}}(E, X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{D}}(E, Y) \xrightarrow{g\circ} \mathrm{Hom}_{\mathcal{D}}(E, Z) \tag{1}$$

and

$$\mathrm{Hom}_{\mathcal{D}}(Z, E) \xrightarrow{\circ g} \mathrm{Hom}_{\mathcal{D}}(Y, E) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{D}}(Z, E) \tag{2}$$

are exact. Lemma 3.14 immediately gives that $\mathrm{im}(f\circ) \subseteq \ker(g\circ)$ and $\mathrm{im}(\circ g) \subseteq \ker(\circ f)$. Thus, we just need to show the other inclusions. Let $\phi \in \mathrm{Hom}_{\mathcal{D}}(E, Y)$ and suppose that $g \circ \phi = 0$. We need to produce a map $h_\phi \in \mathrm{Hom}_{\mathcal{D}}(E, X)$ such that $\phi = f \circ h_\phi$. However, this follows by applying (TR2) and (TR3) in order to obtain h_ϕ as the leftmost dashed arrow in the below morphism of triangles:

$$\begin{tikzcd}
E \arrow[r, "\mathrm{id}"] \arrow[d, dashed, "h_\phi"] & E \arrow[r] \arrow[d, "\phi"] & 0 \arrow[r] \arrow[d] & E[1] \arrow[d, dashed, "h_\phi[1]"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

This proves (1) is exact. Dually, suppose we have $\psi \in \mathrm{Hom}_{\mathcal{D}}(Y, E)$ and $\psi \circ f = 0$. We need to find some $h^\psi \in \mathrm{Hom}_{\mathcal{D}}(Z, E)$ such that $\psi = h^\psi \circ g$. We again use (TR2) and (TR3) to obtain h^ψ as the dashed arrow in the morphism of triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[r] \arrow[d, dashed, "h^\psi"] & X[1] \arrow[d] \\
0 \arrow[r] & E \arrow[r, "\mathrm{id}"] & E \arrow[r] & 0
\end{tikzcd}$$

This proves (2) is exact. ■

Remark 3.16. Proposition 3.15 is particularly important due to how frequently it is used in proofs. It allows us to turn a potentially difficult problem in a (pre-)triangulated category $\mathcal{D}$ into a concrete, comparatively easy problem in the category **Ab** of Abelian groups. More precisely, it makes it possible to relatively easily transfer results from standard homological algebra to analogous results about triangulated categories in the same way that the Yoneda lemma makes it possible to turn set theoretical results into categorical results.

An immediate and incredibly useful application of Proposition 3.15 is the following result, which can be thought of as the triangulated version of the five lemma from homological algebra (see Lemma 2.34 or Corollary 2.35). In particular, the five lemma states that in a the situation of a morphism of two exact sequences of length five, if all component morphisms aside from the middle are known to be isomorphisms, then the middle is also an isomorphism. The triangulated version says this, but for distinguished triangles.

Proposition 3.17. *Let $\mathcal{D}$ be a pre-triangulated category. If, in a morphism*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

of distinguished triangles, the morphisms f and g are isomorphisms, so is h.

Proof. Let $W \in \mathcal{D}$. For notational convenience, we will write $h^W(-)$ for $\mathrm{Hom}_{\mathcal{D}}(W, -)$. By Proposition 3.15, we have a commutative diagram, whose rows are exact, of the form

$$\begin{tikzcd}
h^W(X) \arrow[r] \arrow[d, "f\circ"] & h^W(Y) \arrow[r] \arrow[d, "g\circ"] & h^W(Z) \arrow[r] \arrow[d, dashed, "h\circ"] & h^W(X[1]) \arrow[r] \arrow[d, "f{[1]}\circ"] & h^W(Y[1]) \arrow[d, "g{[1]}\circ"] \\
h^W(X') \arrow[r] & h^W(Y') \arrow[r] & h^W(Z') \arrow[r] & h^W(X'[1]) \arrow[r] & h^W(Y'[1])
\end{tikzcd}$$

in **Ab**, where all solid vertical arrows are isomorphisms. It then follows by the five lemma that $(h\circ)\colon \mathrm{Hom}_{\mathcal{D}}(W, Z) \to \mathrm{Hom}_{\mathcal{D}}(W, Z')$ is an isomorphism for all $W \in \mathcal{D}$, and therefore h is an isomorphism by the Yoneda lemma. ■

Proposition 3.15 also allows us to prove the following lemma, which will be of use later.

Lemma 3.18. *Let $\mathcal{D}$ be a pre-triangulated category. There exists a distinguished triangle*

$$X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow X[1]$$

if and only if the map $f\colon X \to Y$ is an isomorphism.

Proof. ($\Longrightarrow$) Shift the triangle to the right to obtain a morphism of triangles

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\mathrm{id}"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & 0
\end{tikzcd}$$

where all vertical arrows are known to be isomorphisms aside from f. By Proposition 3.17, this implies the dashed arrow, i.e. f, is an isomorphism.

($\Longleftarrow$) If f is an isomorphism, then playing the same argument as above in reverse and using that distinguished triangles are closed under isomorphisms yields the result. ■

As promised in Remark 3.4, we will now explain how to give a precise way in which distinguished triangles give "weak kernel/cokernel pairs." In particular, we first make the following definitions:

Definition 3.19. Let $\mathcal{D}$ be a pre-triangulated category, and let $u\colon X \to Y$ be a morphism. A *weak cokernel* for u is an object $C \in \mathcal{D}$ together with a morphism $v\colon Y \to C$ such that for any $W \in \mathcal{D}$ and map $\phi\colon Y \to W$ such that $\phi \circ u = 0$, there exists some (not necessarily unique!) $\psi\colon C \to W$ such that $\phi = \psi \circ v$, i.e. for any diagram of solid arrows as below, there exists *some* dashed arrow completing it:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, "0"'] & Y \arrow[r, "v"] \arrow[d, "\phi"] & C \arrow[dl, dashed, "\psi"] \\
& W &
\end{tikzcd}$$

A *weak kernel* for u is defined dually.

Using Proposition 3.15, we then immediately have the following proposition, confirming the intuition from Remark 3.4.

Proposition 3.20. *Let $\mathcal{D}$ be a pre-triangulated category, and consider a distinguished triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow X[1].$$

Then Z together with v is a weak cokernel for u and X together with u is a weak kernel for v.

Proof. Let $W \in \mathcal{D}$. Suppose we have a map $\phi : Y \to W$ such that $\phi \circ u = 0$. Then, since $\phi \in \ker(\circ u)$, Proposition 3.15 tells us that there is some $\psi : Z \to W$ such that $\phi = \psi \circ u$. Therefore, Z is a weak cokernel for u.

Dually, let $\psi\colon W \to Y$ be a map such that $v \circ \psi = 0$. Then, since $\psi \in \ker(v\circ)$, Proposition 3.15 tells us that there is some $\phi\colon W \to X$ such that $\psi = u \circ \phi$. Therefore, X is a weak kernel for v. ■

Remark 3.21. One sometimes says that weak (co)kernels that lie in distinguished triangles are *homotopy (co)kernels* for the appropriate morphisms.

Since all (pre-)triangulated categories are additive by definition, they admit finite direct sums. We should hope that this additive structure is respected by the triangulated structure, and in particular, we should hope that the direct sum of two distinguished triangles is again a distinguished triangle. This is true, and will be important later.

Proposition 3.22. *Let $\mathcal{D}$ be a triangulated category. Consider two distinguished triangles*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1].$$

Then the triangle

$$X \oplus X' \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}} Y \oplus Y' \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}} Z \oplus Z' \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}} X[1] \oplus X'[1]$$

is a distinguished triangle.

Proof. By (TR1), we have a distinguished triangle

$$X \oplus X' \longrightarrow Y \oplus Y' \longrightarrow U \longrightarrow X[1] \oplus X'[1].$$

By (TR3), we have maps $Z \to U$, $Z' \to U$ sitting in the commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d] \\
X \oplus X' \arrow[r] & Y \oplus Y' \arrow[r] & U \arrow[r] & X[1] \oplus X'[1] \\
X' \arrow[r] \arrow[u] & Y' \arrow[r] \arrow[u] & Z' \arrow[r] \arrow[u, dashed] & X'[1], \arrow[u]
\end{tikzcd}$$

which induces a map $Z \oplus Z' \to U$. Therefore, we have a diagram

$$\begin{tikzcd}
X \oplus X' \arrow[r] & Y \oplus Y' \arrow[r] & U \arrow[r] & X[1] \oplus X'[1] \\
X \oplus X' \arrow[r] \arrow[u] & Y \oplus Y' \arrow[r] \arrow[u] & Z \oplus Z' \arrow[r] \arrow[u] & X[1] \oplus X'[1]. \arrow[u]
\end{tikzcd}$$

Let $W \in \mathcal{D}$, and for convenience write $X'' = X \oplus X$, $Y'' = Y \oplus Y'$, and $Z'' = Z \oplus Z'$. By Proposition 3.15, $\mathrm{Hom}_{\mathcal{D}}(-, W)$ is cohomological, and therefore—using (TR2)—we obtain a diagram

$$\begin{tikzcd}
\mathrm{Hom}(Y''[1], W) \arrow[r] \arrow[d] & \mathrm{Hom}(X''[1], W) \arrow[r] \arrow[d] & \mathrm{Hom}(U, W) \arrow[r] \arrow[d, dashed] & \mathrm{Hom}(Y'', W) \arrow[r] \arrow[d] & \mathrm{Hom}(X'', W) \arrow[d] \\
\mathrm{Hom}(Y''[1], W) \arrow[r] & \mathrm{Hom}(X''[1], W) \arrow[r] & \mathrm{Hom}(Z'', W) \arrow[r] & \mathrm{Hom}(Y'', W) \arrow[r] & \mathrm{Hom}(X'', W)
\end{tikzcd}$$

in **Ab** where the top row is exact, and all morphisms are isomorphisms aside from the dashed one. The lower row is seen to be exact by commuting the direct sums with Hom and using that the triangles we started with are distinguished, i.e. the bottom row is isomorphic to the direct sum of the exact sequences

$$\mathrm{Hom}(Y[1], W) \longrightarrow \mathrm{Hom}(X[1], W) \longrightarrow \mathrm{Hom}(Z, W) \longrightarrow \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(X, W)$$

and

$$\mathrm{Hom}(Y'[1], W) \longrightarrow \mathrm{Hom}(X'[1], W) \longrightarrow \mathrm{Hom}(Z', W) \longrightarrow \mathrm{Hom}(Y', W) \longrightarrow \mathrm{Hom}(X', W).$$

Then, by the five lemma, the dashed arrow is also an isomorphism, which shows that $\mathrm{Hom}_{\mathcal{D}}(Z'', -) \cong \mathrm{Hom}_{\mathcal{D}}(U, -)$, so $Z \oplus Z' = Z'' \cong U$. Since distinguished triangles are closed under isomorphism, this implies that

$$X \oplus X' \longrightarrow Y \oplus Y' \longrightarrow Z \oplus Z' \longrightarrow X[1] \oplus X'[1].$$

is distinguished. ■

By applying Proposition 3.22 to the distinguished triangles $X \to X \to 0 \to X[1]$ and $0 \to Y \to Y \to 0$, we obtain the following corollary, which will be important, for example, in the proof of Theorem 5.20.

Corollary 3.23. *Let $\mathcal{D}$ be a triangulated category. For any $X, Y \in \mathcal{D}$, the triangle*

$$X \xrightarrow{\iota_X} X \oplus Y \xrightarrow{\pi_Y} Y \xrightarrow{0} X[1]$$

is a distinguished triangle.

Remark 3.24. By induction, Proposition 3.22 extends to cover all finite direct sums of distinguished triangles. However, through an essentially identical proof, it is possible to show that for any indexing set I, if direct sums indexed by I exist, then direct sums of distinguished triangles indexed by I are distinguished. For an explicit proof of this, see [KS06, p. 247, Prop. 10.1.19].

3.2 Issues of Uniqueness

There are various aspects in triangulated categories where uniqueness is not guaranteed. Notably, in (TR3), we do not obtain a unique morphism h in the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

which is also what causes cones to not be functorial. In particular, while (TR3) gives us a weak kind of functoriality and (TR4) gives us distinguished choices satisfying some natural commutative diagram relating f, g, and $g \circ f$, at no point are these choices the only ones. However, it is possible in some situations to have a unique choice (and hence also a unique choice of cone, up to unique isomorphism of triangles).

Here is a simple condition for uniqueness:

Proposition 3.25. *Let $\mathcal{D}$ be a pre-triangulated category and suppose we have two distinguished triangles with morphisms*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, "h"] \arrow[d, dashed, shift right, "\phi"'] \arrow[d, dashed, shift left, "\psi"] & X[1] \arrow[d] \\
X' \arrow[r] & Y' \arrow[r, "f'"] & Z' \arrow[r, "h'"] & X'[1].
\end{tikzcd}$$

where ϕ and ψ are two possible choices of morphisms induced by (TR3). If $\operatorname{Hom}_{\mathcal{D}}(X[1], Z') = 0$ *or* $\operatorname{Hom}_{\mathcal{D}}(Z, Y') = 0$*, then $\phi = \psi$.*

Proof. Given distinguished triangles and morphisms as above, we observe that

$$\phi \circ g = \psi \circ g \quad \text{and} \quad h' \circ \phi = h' \circ \psi$$

so that

$$(\phi - \psi) \circ g = 0 \quad \text{and} \quad h' \circ (\phi - \psi) = 0.$$

By the weak kernel property of $Y' \to Z'$ and the weak cokernel property of $Z \to X[1]$, this implies we have (not necessarily unique) maps $\alpha\colon Z \to Y'$ and $\beta\colon X[1] \to Z'$ such that

$$g' \circ \alpha = \beta \circ h = \phi - \psi.$$

However, if either of the conditions in the proposition statement hold, then this implies

$$\phi - \psi = 0 \implies \phi = \psi$$

as desired. ■

Another, more complicated condition is the following:

Proposition 3.26. *Let $\mathcal{D}$ be a pre-triangulated category, and consider a diagram*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

of solid arrows where the rows are distinguished, and assume further that $\mathrm{Hom}_{\mathcal{D}}(Y, X') = 0$ *and* $\mathrm{Hom}_{\mathcal{D}}(X[1], Y') = 0$*. Then the dashed morphism h induced by (TR3) is unique.*

Proof. It suffices to show that when $f = 0$ and $g = 0$, $h = 0$. In particular, if h_1 and h_2 are choices for h when f, g are arbitrary, then $h_1 - h_2$ will be choices for when $f = g = 0$, so we would have $h_1 - h_2 = 0$ yielding the general result.

Thus, assume $f = g = 0$. By Proposition 3.20, $X[1]$ together with w is a weak cokernel for v, and Y' together with v' is a weak kernel for w'. We have that $w' \circ h = 0$, so since Y' is a weak kernel there is some $p \colon Z \to Y'$ such that $h = v' \circ p$. Similarly, $h \circ v = 0$ so $X[1]$ being a weak cokernel tells us that there is some $q \colon X[1] \to Z'$ such that $h = q \circ w$. We therefore have the following (not necessarily commutative) diagram of solid arrows:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "0"] & Y \arrow[r, "v"] \arrow[d, "0"] \arrow[dl, dashed, "r"'] & Z \arrow[r, "w"] \arrow[d, "h"] \arrow[dl, "p"'] & X[1] \arrow[d, "0"] \arrow[dl, "q"'] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

We conclude the existence of the dashed arrow $r \colon Y \to X'$ using (TR2) and (TR3), by placing it in the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, dashed, "r"] & Z \arrow[r, "w"] \arrow[d, "p"] & X[1] \arrow[r, "-u'[1]"] \arrow[d, "q"] & Y[1] \arrow[d, dashed, "r[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

By assumption, $r = 0$ since it is a map $Y' \to X$. However, by commutativity, this implies that

$$p \circ v = u' \circ r = u' \circ 0 = 0$$

and since $X[1]$ is a weak cokernel, this means that there is a map $s : X[1] \to Y'$ such that $p = s \circ w$. However, by assumption we have that $\mathrm{Hom}_{\mathcal{D}}(X[1], Y') = 0$, so $s = 0$. Therefore, $p = 0$, and since $h = v' \circ p$ we get that $h = 0$. ■

Corollary 3.27. *Let $\mathcal{D}$ be a pre-triangulated category, let $X, Y \in \mathcal{D}$, and let $u \colon X \to Y$ be a morphism. Suppose that* $\mathrm{Hom}_{\mathcal{D}}(Y, X) = 0$ *and* $\mathrm{Hom}_{\mathcal{D}}(X[1], Y) = 0$*. Then the cone C_u in the distinguished triangle*

$$X \xrightarrow{u} Y \longrightarrow C_u \longrightarrow X[1]$$

is unique up to unique isomorphism of triangles.

Proof. If Z is any other choice of cone, then it is clear that it sits in a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & C_u \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r] & Z \arrow[r] & X[1]
\end{tikzcd}$$

where all vertical arrows are isomorphisms. However, the assumptions allow us to deduce (using the above proposition) that this isomorphism is unique. ■

The above gives a criterion which guarantees uniqueness of the cone construction in select circumstances. There are other ways in which non-uniqueness arises in triangulated categories, however. For example, consider a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

We can wonder if there any other distinguished triangles which contain the sequence

$$X \longrightarrow Y \longrightarrow Z,$$

or rather, if the morphism $Z \to X[1]$ is unique in this regard. In general, we have no way of knowing this. However, with a relatively mild assumption, we can obtain the following lemma (which will be used later in the proofs of several results in Section 5):

Lemma 3.28. *Let $\mathcal{D}$ be a triangulated category, and consider two distinguished triangles*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h_i} X[1]$$

$i = 1, 2$. *If* $\operatorname{Hom}_{\mathcal{D}}(X[1], Z) = 0$, *then* $h_1 = h_2$.

Proof. By (TR3), we have some morphism $\phi\colon Z \to Z$ such that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h_1"] \arrow[d, dashed, "\phi"] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h_2"] & X[1]
\end{tikzcd}$$

commutes. Therefore, we have that $h_1 = h_2 \circ \phi$ and $\phi \circ g = g$, so $(\mathrm{id} - \phi) \circ g = 0$. By Proposition 3.20, $X[1]$ together with h_1 is a weak cokernel for g, and therefore there exists some $\psi\colon X[1] \to Z$ such that $\psi \circ h_1 = \mathrm{id} - \phi$. By assumption, $\psi = 0$, and hence $\phi = \mathrm{id}$, from which we obtain that $h_1 = h_2$. ■

3.3 LOCALIZATION OF CATEGORIES

It is useful, in certain contexts, to impose upon a category additional relations that cause previously non-invertible morphisms to become invertible. The most obvious example of this will be expounded upon in Section 4, where we construct *derived categories,* i.e. categories formed by formally inverting quasi-isomorphisms in categories of chain complexes.

Definition 3.29. Let $\mathcal{C}$ be a category, and let $\mathcal{S}$ be a class of morphisms in $\mathcal{C}$. A (big) category $\mathcal{D}$ together with a functor $Q\colon \mathcal{C} \to \mathcal{D}$ is a *(strict) localization of $\mathcal{C}$ at $\mathcal{S}$* if

(i) for all morphisms $f \in \mathcal{S}$, the morphism $Q(f)$ is an isomorphism in $\mathcal{D}$,

(ii) for all (big) categories $\mathcal{E}$ with a functor $G : \mathcal{C} \to \mathcal{E}$ which sends morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{E}$, there is a unique functor $G'\colon \mathcal{D} \to \mathcal{E}$ such that $G = G' \circ Q$.

When such a localization exists, we denote it by $\mathcal{C}_{\mathcal{S}}$, and the functor G' above is denoted by $G_{\mathcal{S}}$.

Remark 3.30. There is also a notion of "weak" localization, where one replaces the above 1-categorical definition with the appropriate 2-categorical version, and this is the definition which [KS06] uses. We take the simpler 1-categorical definition because it will be more convenient for our purposes. This will not be a problem, since a strict localization is automatically a weak localization. In particular, the localization as defined above also satisfies the following stronger version of (ii): the functor Q induces an isomorphism of categories

$$(\circ Q)\colon \mathrm{Fun}(\mathcal{C}_{\mathcal{S}}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}_{\mathcal{S}}(\mathcal{C}, \mathcal{E})$$

where $\mathrm{Fun}_{\mathcal{S}}(\mathcal{C}, \mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ consisting of functors sending morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{E}$. In a weak localization, this is merely an equivalence.

Remark 3.31. When a localization exists, it is unique up to *isomorphism* of categories. Note that in the 2-categorical case, this is not true: a weak localization is only defined up to *equivalence* of categories.

Remark 3.32. Even when $\mathcal{C}$ is a locally small category, the localization $\mathcal{C}_{\mathcal{S}}$ does not need to be locally small.

In general, the localization $\mathcal{C}_{\mathcal{S}}$ is complicated to describe. However, if we place some restrictions on $\mathcal{S}$, then one can give a relatively simple description. To give some motivation first, note that in the situation of (for example) rings, localization can also be defined essentially as above, using a universal property. However, it is not until we impose that the localizing class is *multiplicative* that one has a good description in terms of fractions. Thus, we want to have the same thing for categories.

Definition 3.33. Let $\mathcal{C}$ be a category, and let $\mathcal{S}$ be a class of morphisms in $\mathcal{C}$. The class $\mathcal{S}$ is a *right multiplicative system* if it satisfies the following criteria:

(M1) any isomorphism in $\mathcal{C}$ is also in $\mathcal{S}$,

(M2) for any two morphisms $(f\colon X \to Y), (g\colon Y \to Z) \in \mathcal{S}$, we also have $g \circ f \in \mathcal{S}$,

(M3) given a morphism $f : X \to Y$ in $\mathcal{C}$ and a morphism $s : X \to X'$ in $\mathcal{S}$, there is some $t\colon Y \to Y'$ in $\mathcal{S}$ and $g\colon X' \to Y'$ which fit into a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "\exists t"] \\ X' \arrow[r, dashed, "\exists g"] & Y' \end{tikzcd}$$

(M4) for any two parallel morphisms $f, g\colon X \to Y$ in $\mathcal{C}$, if there exists a morphism $s\colon W \to X$ in $\mathcal{S}$ such that $f \circ s = g \circ s$, then there exists a morphism $t\colon Y \to Z$ in $\mathcal{S}$ such that $t \circ f = t \circ g$, i.e. we have the commutative diagram

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, dashed, "\exists t"] & Z. \end{tikzcd}$$

A class of morphisms $\mathcal{S}$ is a *left multiplicative system* if $\mathcal{S}^{\mathrm{op}}$ is a right multiplicative system in $\mathcal{C}^{\mathrm{op}}$. A class of morphisms is a *multiplicative system* if it is left multiplicative and right multiplicative.

Remark 3.34. The above definition follows the naming scheme of [KS06, p. 151–152, Definition 7.1.5]. In a remark (specifically, Remark 7.1.7), the authors warn that this terminology is not universal. In particular, some resources swap the names (so that our left multiplicative systems are their right multiplicative systems).

We will now define several categories of interest.

Definition 3.35. Let $\mathcal{C}$ be a category, and let $\mathcal{S}$ be a class of morphisms satisfying (M1) and (M2) above. Fix some $X \in \mathcal{C}$. Define the categories $\mathcal{S}_{/X}$, $\mathcal{S}^{X/}$, and functors $\pi_{/X} : \mathcal{S}_{/X} \to \mathcal{C}$, $\pi^{X/} \colon \mathcal{S}^{X/} \to \mathcal{C}$ as follows:

$$\mathcal{S}_{/X} := \{s \colon X' \to X \mid s \in \mathcal{S}\},$$

$$\mathrm{Hom}_{\mathcal{S}_{/X}}((s\colon X' \to X),(t\colon X'' \to X)) := \left\{ f \in \mathrm{Hom}_{\mathcal{C}}(X', X'') \;\middle|\; \begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr, "s"'] & & X'' \arrow[dl, "t"] \\ & X & \end{tikzcd} \text{ commutes} \right\},$$

$$\mathcal{S}^{X/} := \{s \colon X \to X' \mid s \in \mathcal{S}\},$$

$$\mathrm{Hom}_{\mathcal{S}^{X/}}((s\colon X \to X'),(t\colon X \to X'')) := \left\{ f \in \mathrm{Hom}_{\mathcal{C}}(X', X'') \;\middle|\; \begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X' \arrow[rr, "f"] & & X'' \end{tikzcd} \text{ commutes} \right\},$$

$$\pi_{/X}(s\colon X' \to X) := X',$$
$$\pi^{X/}(s\colon X \to X') := X'.$$

Remark 3.36. Here, our notation differs from [KS06]. In particular, what we call $\mathcal{S}^{X/}$ and $\mathcal{S}_{/X}$, they call $\mathcal{S}^X$ and $\mathcal{S}_X$. They also refer to the projection functors $\pi^{X/}$ and $\pi_{/X}$ as α^X and α_X.

In the situation that $\mathcal{S}$ is a (left/right) multiplicative system, we will use the above categories to make a workable construction of the localization $\mathcal{C}_{\mathcal{S}}$. In particular, for $\mathcal{S}$ a right (resp. left) multiplicative system, we will define a category $\mathcal{C}^r_{\mathcal{S}}$ (resp. $\mathcal{C}^l_{\mathcal{S}}$), which will give the localization.

Definition 3.37. Let $\mathcal{C}$ be a category, and let $\mathcal{S}$ be a right multiplicative system. For $* \in \{l, r\}$, let the objects of $\mathcal{C}^*_{\mathcal{S}}$ simply be the objects of $\mathcal{C}$. Define the Hom-sets

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X,Y) := \varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \pi^{Y/}(-)) = \varinjlim_{(Y \to Y') \in \mathcal{S}^{Y/}} \mathrm{Hom}_{\mathcal{C}}(X, Y').$$

When $\mathcal{S}$ is instead a left multiplicative system, define the Hom-sets

$$\mathrm{Hom}_{\mathcal{C}^l_{\mathcal{S}}}(X,Y) := \varinjlim \mathrm{Hom}_{\mathcal{C}}(\pi_{/X}(-), Y) = \varinjlim_{(X' \to X) \in \mathcal{S}_{/X}} \mathrm{Hom}_{\mathcal{C}}(X', Y).$$

We want to make sure that this defines an actual category (note that we need to define composition). To do this, we will explicitly compute these colimits. To do that, we first make the observation that whenever $\mathcal{S}$ is a right (resp. left) multiplicative system, the category $\mathcal{S}^{X/}$ (resp. $\mathcal{S}_{/X}$) is filtered (resp. cofiltered; see [KS06, p. 72, Def. 3.1.1]) for any $X \in \mathcal{C}$.

Proposition 3.38. *Let $\mathcal{C}$ be a category, and let $\mathcal{S}$ be a right (resp. left) multiplicative system. Then, for any $X \in \mathcal{C}$, the category $\mathcal{S}^{X/}$ (resp. $\mathcal{S}_{/X}$) is filtered (resp. cofiltered).*

Proof. We prove only the case when $\mathcal{S}$ is a right multiplicative system, since the proof in the other case is dual.

Certainly, $\mathcal{S}^{X/}$ is non-empty, since $\mathrm{id}_X \in \mathcal{S}^{X/}$ by (M1). Now let

$$(x' \colon X \to X'), (x'' \colon X \to X'') \in \mathcal{S}^{X/}.$$

By (M3), we then find that there exists an object X''' and morphisms $s \colon X' \to X'''$, $g \colon X'' \to X'''$ with $s \in \mathcal{S}$ fitting into the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "x'"] \arrow[d, "x''"'] & X' \arrow[d, dashed, "s"] \\
X'' \arrow[r, dashed, "g"] & X'''
\end{tikzcd}$$

and by (M2), we have $s \circ x' \in \mathcal{S}$, hence $(X \xrightarrow{s \circ x'} X''') \in \mathcal{S}^{X/}$.

Finally, consider two parallel arrows $f, g \colon x' \to x''$, i.e. maps in $\mathcal{C}$ fitting into the commutative triangle

$$\begin{tikzcd}
 & X \arrow[dl, "x'"'] \arrow[dr, "x''"] & \\
X' \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & X''
\end{tikzcd}$$

We then observe that (M4) provides us with an object X''' and a morphism $t \colon X'' \to X'''$ which makes the diagram

$$\begin{tikzcd}
X \arrow[r, "x'"] & X' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X'' \arrow[r, dashed, "t"] & X'''
\end{tikzcd}$$

commute. Since $x'' \in \mathcal{S}$, we have $t \circ x'' \in \mathcal{S}$, which makes $x''' := t \circ x''$ an element of $\mathcal{S}^{X/}$. The map $t \colon x'' \to x'''$ then satisfies the desired requirement that $t \circ f = t \circ g$. ■

When the indexing category of a colimit of sets is filtered, we have a good description of it. In particular, we get the following:

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, Y) = \{(Y', s, f) \mid (s \colon Y \to Y') \in \mathcal{S}^{Y/},\ f \colon X \to Y'\}/\sim$$

where $\sim$ is the equivalence relation given by $(Y', s, f) \sim (Y'', t, g)$ if and only if there exists $(Y \xrightarrow{u} Y''') \in \mathcal{S}^{Y/}$, $q' \colon Y' \to Y'''$, $q'' \colon Y'' \to Y'''$, fitting into a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "s"'] \arrow[d, dashed, "u"] \arrow[dr, "t"] & \\
Y' \arrow[r, dashed, "q'"] & Y''' & Y'' \arrow[l, dashed, "q''"'] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

which, one notes, also defines an element $(Y''', u, h) \in \mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, Y)$, where $h = q' \circ f = q'' \circ g$.

The situation for left multiplicative systems is similar. In particular, one sees that

$$\mathrm{Hom}_{\mathcal{C}^l_{\mathcal{S}}}(X, Y) = \{(X', s, f) \mid (s \colon X' \to X) \in \mathcal{S}_{/X},\ f \colon X' \to Y\}/\sim$$

where $\sim$ is the equivalence relation given by $(X', s, f) \sim (X'', t, g)$ if and only if there exists some $(X''' \xrightarrow{u} X) \in \mathcal{S}_{/X}$ together with maps $r' \colon X''' \to X'$, $r'' \colon X''' \to X''$ fitting into a commutative diagram

$$\begin{tikzcd}
& Y & \\
X' \arrow[ur, "f"] \arrow[dr, "s"'] & X''' \arrow[l, dashed, "r'"'] \arrow[r, dashed, "r''"] \arrow[d, dashed, "u"'] & X'' \arrow[ul, "g"'] \arrow[dl, "t"] \\
& X &
\end{tikzcd}$$

Similarly, this diagram too defines an element of $\mathrm{Hom}_{\mathcal{C}^l_{\mathcal{S}}}(X, Y)$ given by (X''', u, h), where $h = f \circ r' = g \circ r''$.

We now define the composition. Consider a right multiplicative system $\mathcal{S}$, and let $(Y', s, f) \in \mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, Y)$, $(Z', t, g) \in \mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(Y, Z)$. We then have a diagram

$$\begin{tikzcd}
& Y' & & Z' & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"] \arrow[ur, "g"] & & Z \arrow[ul, "t"]
\end{tikzcd}$$

and using (M3) on the maps s, g, we obtain some object W along with maps $h : Y' \to W$, $u \colon Z' \to W$, with $u \in \mathcal{S}$ fitting into the following commutative diagram

$$\begin{tikzcd}
& & W & & \\
& Y' \arrow[ur, dashed, "h"] & & Z' \arrow[ul, dashed, "u"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"']
\end{tikzcd}$$

and we set

$$(Y', s, f) \circ (Z', t, g) := (W, h \circ f, u \circ t).$$

This gives a well-defined notion of composition. Checking that this is well-defined is essentially elementary, but requires drawing quite a large diagram so we omit it. The situation for a left multiplicative system is entirely dual, so we also skip writing that down. Thus, we have defined a category $\mathcal{C}^r_{\mathcal{S}}$ (when $\mathcal{S}$ is right multiplicative; $\mathcal{C}^l_{\mathcal{S}}$ when $\mathcal{S}$ is left multiplicative). When $\mathcal{S}$ is both left and right multiplicative, one may use (M3) to give maps

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, Y) \rightleftarrows \mathrm{Hom}_{\mathcal{C}^l_{\mathcal{S}}}(X, Y)$$

which are natural bijections, so that in that situation we obtain an equivalence

$$\mathcal{C}^r_{\mathcal{S}} \cong \mathcal{C}^l_{\mathcal{S}}.$$

More explicitly, consider a morphism $X \xrightarrow{f} Y' \xleftarrow{s} Y$ in $\mathcal{C}^r_{\mathcal{S}}$. Using (M3), but now for the *left* case, we then obtain the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y' \\
X' \arrow[u, dashed, "s'"'] \arrow[r, dashed, "f'"] & Y \arrow[u, "s"']
\end{tikzcd}$$

which defines a morphism (X', s', f') in $\mathcal{C}^l_{\mathcal{S}}$. The other direction is dual. That this gives a natural bijection

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X,Y) \cong \mathrm{Hom}_{\mathcal{C}^l_{\mathcal{S}}}(X,Y)$$

is clear.

Remark 3.39. One may show much of the above more systematically by defining the composition using universal properties, as is done in [KS06]. We opt for the above approach since the precise details are not of too much interest to us.

Notation 3.40. Consider a morphism

$$X \xrightarrow{f} Y' \xleftarrow{s} Y$$

in $\mathcal{C}^r_{\mathcal{S}}$. We will sometimes write $s^{-1}f$ to denote this, particularly when the maps f and s are clear. Similarly, a morphism

$$X \xleftarrow{s} X' \xrightarrow{f} Y$$

in $\mathcal{C}^l_{\mathcal{S}}$ will sometimes be denoted fs^{-1}.

Definition 3.41. Let $\mathcal{C}$ be a category, and let $\mathcal{S}$ be a right (resp. left) multiplicative system. Then we define the *localization functor*

$$Q = Q^r_{\mathcal{S}} \colon \mathcal{C} \to \mathcal{C}^r_{\mathcal{S}} \quad (\text{resp. } Q^l_{\mathcal{S}} \colon \mathcal{C} \to \mathcal{C}^l_{\mathcal{S}})$$

by $Q^r_{\mathcal{S}}(X) = X$, and $Q^r_{\mathcal{S}}(X \xrightarrow{f} Y) = (Y, \mathrm{id}_Y, f)$ (resp. $Q^l_{\mathcal{S}}(X) = X$, $Q^l_{\mathcal{S}}(f) = (X, \mathrm{id}_X, f)$). When no confusion is possible, we write $Q = Q^r_{\mathcal{S}} = Q^l_{\mathcal{S}}$.

Pictorially, $Q^r_{\mathcal{S}}$ sends a morphism $f \colon X \to Y$ to the diagram

$$X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y} Y.$$

It can be checked that when $\mathcal{S}$ is a multiplicative system, the equivalence $\mathcal{C}^r_{\mathcal{S}} \cong \mathcal{C}^l_{\mathcal{S}}$ briefly described is compatible with the functors $Q^r_{\mathcal{S}}$ and $Q^l_{\mathcal{S}}$ in the sense that the diagram

$$\begin{tikzcd}
 & \mathcal{C}^r_{\mathcal{S}} \arrow[dd, shift left] \\
\mathcal{C} \arrow[ur, "Q^r_{\mathcal{S}}"] \arrow[dr, "Q^l_{\mathcal{S}}"'] & \\
 & \mathcal{C}^l_{\mathcal{S}} \arrow[uu, shift left]
\end{tikzcd}$$

commutes strictly. Indeed, denoting by $F : \mathcal{C}^r_{\mathcal{S}} \to \mathcal{C}^l_{\mathcal{S}}$ and $G : \mathcal{C}^l_{\mathcal{S}} \to \mathcal{C}^r_{\mathcal{S}}$ the above described equivalences, it is easily seen that

$$\begin{aligned}
(F \circ Q^r_{\mathcal{S}})(X \xrightarrow{f} Y) &= F(X \xrightarrow{f} Y \xleftarrow{\mathrm{id}} Y) \\
&= (X, \mathrm{id}, f), \\
(G \circ Q^l_{\mathcal{S}})(X \xrightarrow{f} Y) &= G(X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y) \\
&= (Y, \mathrm{id}, f)
\end{aligned}$$

so that $F \circ Q^r_{\mathcal{S}} = Q^l_{\mathcal{S}}$ and $G \circ Q^l_{\mathcal{S}} = Q^r_{\mathcal{S}}$.

The way to think about the localization $\mathcal{C}^r_{\mathcal{S}}$ is that the diagram

$$X \xrightarrow{f} Y' \xleftarrow{s} Y$$

represents the composition "$s^{-1} \circ f$" even though this may not exist. However, we will see that it does make clear the fact that it describes the composition

$$Q(s)^{-1} \circ Q(f).$$

We will now check that these categories are actually localizations.

Proposition 3.42. *Let $\mathcal{C}$ be a category, and let $\mathcal{S}$ be a right (resp. left) multiplicative system. Then $\mathcal{C}^r_{\mathcal{S}}$ (resp. $\mathcal{C}^l_{\mathcal{S}}$) together with $Q^r_{\mathcal{S}}$ (resp. $Q^l_{\mathcal{S}}$) is a localization of $\mathcal{C}$ at $\mathcal{S}$. In particular, any morphism*

$$X \xrightarrow{f} Y' \xleftarrow{s} Y$$

in $\mathcal{C}^r_{\mathcal{S}}$ can be written as $Q(s)^{-1} \circ Q(f)$.

Proof. We show only the case when $\mathcal{S}$ is right multiplicative. First we show that morphisms in $\mathcal{S}$ are sent to isomorphisms by Q. Consider a map $s\colon X \to Y$ in $\mathcal{S}$, and note that $Q(s) = (Y, \mathrm{id}_Y, s)$. Let $t\colon Q(Y) \to Q(X)$ be the morphism in $\mathcal{C}^r_{\mathcal{S}}$ defined by (Y, s, id_X). Then

$$t \circ Q(s) = (X \xrightarrow{s} Y \xleftarrow{s} X), \quad Q(s) \circ t = (Y \xrightarrow{\mathrm{id}} Y \xleftarrow{\mathrm{id}} X) = \mathrm{id}_{Q(Y)}.$$

Thus it remains to show that $(Y, s, s) = \mathrm{id}_{Q(X)}$. However, this is clear from the following diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "s"'] \arrow[d, "s"] \arrow[dr, "\mathrm{id}"] & \\
Y \arrow[r, "\mathrm{id}"] & Y & X \arrow[l, "s"'] \\
 & X \arrow[ul, "s"] \arrow[u, "s"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

Thus, Q sends morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{C}^r_{\mathcal{S}}$.

Consider now an arbitrary category $\mathcal{E}$ along with a functor $G\colon \mathcal{C} \to \mathcal{E}$ such that $G(s)$ is an isomorphism for all $s \in \mathcal{S}$. Define a functor $G_{\mathcal{S}}\colon \mathcal{C}^r_{\mathcal{S}} \to \mathcal{E}$ as follows:

$$G_{\mathcal{S}}(X) := G(X), \quad G_{\mathcal{S}}(X \xrightarrow{f} Y' \xleftarrow{s} Y) := G(s)^{-1} \circ G(f).$$

This is well-defined. In particular, suppose we have two representatives (Y', s, f), (Y'', s', f') of the map given above. Them being equivalent yields a third representative (Y''', s'', f'') fitting into the diagram

$$\begin{tikzcd}
 & Y' \arrow[d, "q"] & \\
X \arrow[ur, "f"] \arrow[r, "f''"] \arrow[dr, "f'"'] & Y''' & Y \arrow[ul, "s"'] \arrow[l, "s''"'] \arrow[dl, "s'"] \\
 & Y'' \arrow[u, "q'"'] &
\end{tikzcd}$$

from which we calculate

$$\begin{aligned} G(s)^{-1} \circ G(f) &= G(s'')^{-1} \circ G(q) \circ G(f) \\ &= G(s'')^{-1} \circ G(f'') \\ &= G(s'')^{-1} \circ G(q') \circ G(f') \\ &= G(s')^{-1} \circ G(f'). \end{aligned}$$

Thus, $G_{\mathcal{S}}$ does not depend on the representative chosen for a morphism in $\mathcal{C}^r_{\mathcal{S}}$. Furthermore, if we have two morphisms

$$X \xrightarrow{f} Y' \xleftarrow{s} Y, \quad Y \xrightarrow{g} Z' \xleftarrow{t} Z$$

then the diagram

$$\begin{tikzcd} & & W & & \\ & Y' \arrow[ur, "h"] & & Z' \arrow[ul, "u"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"'] \end{tikzcd}$$

defining the composition tells us that

$$\begin{aligned} G_{\mathcal{S}}((Z',t,g) \circ (Y',s,f)) &= G(u \circ t)^{-1} \circ G(h \circ f) \\ &= G(t)^{-1} \circ G(u)^{-1} \circ G(h) \circ G(f) \\ &= G(t)^{-1} \circ G(g) \circ G(s)^{-1} \circ G(f) \\ &= G_{\mathcal{S}}((Z',t,g)) \circ G_{\mathcal{S}}((Y',s,f)) \end{aligned}$$

so $G_{\mathcal{S}}$ does define a functor. It is immediately clear that $G_{\mathcal{S}} \circ Q = G$. It remains to check that $G_{\mathcal{S}}$ is unique. Suppose we have another functor $G' \colon \mathcal{C}_{\mathcal{S}} \to \mathcal{E}$ such that $G' \circ Q = G$. Then it is clear that $G_{\mathcal{S}}$ and G' agree on objects. Furthermore, for any morphism $s^{-1}f \colon X \xrightarrow{f} Y' \xleftarrow{s} Y$ in $\mathcal{C}_{\mathcal{S}}$, we know that G' must satisfy $G'(s^{-1}f) = G'(Q(s)^{-1} \circ Q(f)) = G(s)^{-1} \circ G(f) = G_{\mathcal{S}}(s^{-1}f)$, and therefore $G' = G_{\mathcal{S}}$. ■

Thus we see that the category $\mathcal{C}^r_{\mathcal{S}}$ is the localization, and we may happily denote it $\mathcal{C}_{\mathcal{S}}$. The upside of having done all the above work is that we now have an explicitly workable theory of localization which we may then apply to triangulated categories. Shortly, when we involve triangulated categories, we will need that localizations of additive categories are additive, and that the localization functor is an additive functor. Showing this requires some theorems in abstract nonsense. Our strategy will be to show that localization is well-behaved with respect to products and adjoints.

If we have categories $\mathcal{C}$ and $\mathcal{D}$ equiped with classes of morphisms $\mathcal{S}$ and $\mathcal{T}$, then a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that $F(\mathcal{S}) \subseteq \mathcal{T}$ induces a functor $\mathcal{C}_{\mathcal{S}} \to \mathcal{D}_{\mathcal{T}}$ (in particular, by applying condition (ii) from the definition of localization to $Q_{\mathcal{T}} \circ F$). In particular, this lies in the commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q_{\mathcal{S}}"'] & \mathcal{D} \arrow[d, "Q_{\mathcal{T}}"] \\ \mathcal{C}_{\mathcal{S}} \arrow[r, dashed] & \mathcal{D}_{\mathcal{T}} \end{tikzcd}$$

We need the following:

Proposition 3.43. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{S}$ (resp. $\mathcal{T}$) be a class of morphisms in $\mathcal{C}$ (resp. $\mathcal{D}$). Suppose we have functors $L\colon \mathcal{C} \to \mathcal{D}$, $R\colon \mathcal{D} \to \mathcal{C}$ such that L is left adjoint to R and $L(\mathcal{S}) \subseteq \mathcal{T}$, $R(\mathcal{T}) \subseteq \mathcal{S}$. Denote by $L_{\mathcal{S}}\colon \mathcal{C}_{\mathcal{S}} \to \mathcal{D}_{\mathcal{T}}$, $R_{\mathcal{T}}\colon \mathcal{D}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ the induced functors described above. Then $L_{\mathcal{S}}$ is left adjoint to $R_{\mathcal{T}}$.*

Proof. We will use the characterization of adjoints as absolute Kan extensions (see [Rie17, Ch. 6], especially Prop. 6.5.2). In particular, we will show that $R_{\mathcal{T}} = \operatorname{Lan}_{L_{\mathcal{S}}} \operatorname{id}_{\mathcal{C}_{\mathcal{S}}}$ and that this holds absolutely, i.e. that $G \circ R_{\mathcal{T}} = \operatorname{Lan}_{L_{\mathcal{S}}}(G)$. We need to show that we have a natural isomorphism

$$\operatorname{Hom}(G, F \circ L_{\mathcal{S}}) = \operatorname{Hom}(G \circ R_{\mathcal{T}}, F)$$

for all categories $\mathcal{E}$ with functors $G\colon \mathcal{C}_{\mathcal{S}} \to \mathcal{E}$, $F: \mathcal{D}_{\mathcal{T}} \to \mathcal{E}$. Since L is left adjoint to R (i.e. R is right adjoint to L), we know that these satisfy analogous conditions. Therefore, we have the following natural isomorphisms:

$$\begin{aligned}
\operatorname{Hom}(G, F \circ L_{\mathcal{S}}) &\cong \operatorname{Hom}(G \circ Q_{\mathcal{S}}, F \circ L_{\mathcal{S}} \circ Q_{\mathcal{S}}) \\
&\cong \operatorname{Hom}(G \circ Q_{\mathcal{S}}, F \circ Q_{\mathcal{T}} \circ L) \\
&\cong \operatorname{Hom}(G \circ Q_{\mathcal{S}} \circ R, F \circ Q_{\mathcal{T}}) \\
&\cong \operatorname{Hom}(G \circ R_{\mathcal{T}} \circ Q_{\mathcal{T}}, F \circ Q_{\mathcal{T}}) \cong \operatorname{Hom}(G \circ R_{\mathcal{T}}, F)
\end{aligned}$$

where two of these follow by the definition of localization (in particular, the fact that strict localizations are also weak localizations), two follow by commutativity, and one follows by the fact that L is left adjoint to R. This shows the desired isomorphism. ■

The above proposition essentially says that localization preserves adjoints. We now wish to show that (finite) products of localizations are the same as localizations of finite products.

Proposition 3.44. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{S}$ (resp. $\mathcal{T}$) be a class of morphisms in $\mathcal{C}$ (resp. $\mathcal{D}$). Then there is a canonical isomorphism of categories*

$$(\mathcal{C} \times \mathcal{D})_{\mathcal{S} \times \mathcal{T}} \cong \mathcal{C}_{\mathcal{S}} \times \mathcal{D}_{\mathcal{T}}.$$

Proof. For any category $\mathcal{E}$, we have natural *isomorphisms (!)* of categories

$$\begin{aligned}
\operatorname{Fun}(\mathcal{C}_{\mathcal{S}} \times \mathcal{D}_{\mathcal{T}}, \mathcal{E}) &\cong \operatorname{Fun}(\mathcal{C}_{\mathcal{S}}, \operatorname{Fun}(\mathcal{D}_{\mathcal{T}}, \mathcal{E})) \\
&\cong \operatorname{Fun}_{\mathcal{S}}(\mathcal{C}, \operatorname{Fun}(\mathcal{D}_{\mathcal{T}}, \mathcal{E})) \\
&\cong \operatorname{Fun}_{\mathcal{S}}(\mathcal{C}, \operatorname{Fun}_{\mathcal{T}}(\mathcal{D}, \mathcal{E})) \\
&\cong \operatorname{Fun}_{\mathcal{S} \times \mathcal{T}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \operatorname{Fun}((\mathcal{C} \times \mathcal{D})_{\mathcal{S} \times \mathcal{T}}, \mathcal{E})
\end{aligned}$$

where we use that the product is left adjoint to Fun in the category **Cat** of categories. The fourth isomorphism can be checked by explicitly writing out the adjunction map and checking how it affects isomorphisms. We now apply the Yoneda lemma to deduce the result. ■

We will now show that localizations of additive categories are additive. The idea is that the existence of finite products is equivalent to the existence of a left adjoint of the diagonal functor

$$\mathcal{C} \to \mathcal{C}^n$$

for all finite n.

Lemma 3.45. *Let $\mathcal{C}$ be a category, and let $\mathcal{S}$ be a right (resp. left) multiplicative system. Then the localization functor $Q\colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ commutes with finite colimits (resp. limits).*

Proof. We assume $\mathcal{S}$ is a right multiplicative system, since the other case is dual. Let I be a finite category, and let $D\colon I \to \mathcal{C}$ be a diagram. Assume that $\varinjlim_{i\in I} D(i)$ exists. For any $Y \in \mathcal{C}_{\mathcal{S}}$, we then have natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}_{\mathcal{S}}}(Q(\varinjlim_i D(i)), Y) &= \varinjlim_{(Y\to Y')\in \mathcal{S}^{Y/}} \operatorname{Hom}_{\mathcal{C}}(\varinjlim_i D(i), Y') \\ &\cong \varinjlim_{(Y\to Y')\in \mathcal{S}^{Y/}} \varprojlim_i \operatorname{Hom}_{\mathcal{C}}(D(i), Y') \\ &\cong \varprojlim_i \varinjlim_{(Y\to Y')\in \mathcal{S}^{Y/}} \operatorname{Hom}_{\mathcal{C}}(D(i), Y') \\ &= \varprojlim_i \operatorname{Hom}_{\mathcal{C}_{\mathcal{S}}}(Q(D(i)), Y)\end{aligned}$$

where the third row follows because finite limits commute with filtered colimits in **Set**. Therefore, $Q(\varinjlim_{i\in I} D(i))$ represents the desired colimit. ■

Proposition 3.46. *Let $\mathcal{C}$ be an additive category, and let $\mathcal{S}$ be a multiplicative system. Then $\mathcal{C}_{\mathcal{S}}$ is additive and the localization functor $Q\colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ is additive.*

Proof. Since $\mathcal{C}$ admits finite products, there is a right adjoint $P^{(n)}$ to the diagonal functor $\Delta_{\mathcal{C}}^{(n)}\colon \mathcal{C} \to \mathcal{C}^n$ for all $n \geq 0$, given by taking the product of n objects. Similarly, since $\mathcal{C}$ admits finite coproducts, there is a left adjoint $S^{(n)}$ to $\Delta_{\mathcal{C}}^{(n)}$ for all $n \geq 0$, given by taking the coproduct of n objects. Note that $(\Delta_{\mathcal{C}}^{(n)})_{\mathcal{S}} = \Delta_{\mathcal{C}_{\mathcal{S}}}^{(n)}$, and therefore $\mathcal{C}_{\mathcal{S}}$ admits finite products and coproducts by Proposition 3.43. Since Q commutes with finite limits and colimits by Lemma 3.45, it follows that finite products and finite coproducts agree in $\mathcal{C}_{\mathcal{S}}$. In particular, every object in $\mathcal{C}_{\mathcal{S}}$ is of the form $Q(X)$ for some $X \in \mathcal{C}$, and we have canonical isomorphisms

$$Q(X) \sqcup Q(Y) \xrightarrow{\sim} Q(X \sqcup Y) \xrightarrow{\sim} Q(X \times Y) \xrightarrow{\sim} Q(X) \times Q(Y).$$

Since Q preserves finite limits and colimits, it preserves finite products and coproducts. Therefore, since in addition Q is essentially surjective, by Proposition 2.12, $\mathcal{C}_{\mathcal{S}}$ inherits an additive structure such that Q is additive. ■

3.4 THE VERDIER QUOTIENT

Suppose we have a triangulated category $\mathcal{D}$. In the situation that a multiplicative system $\mathcal{S}$ on $\mathcal{D}$ has no relation to the triangulated structure, there is no way of knowing whether the localization $\mathcal{D}_{\mathcal{S}}$ has a reasonable triangulated structure on it which makes the localization functor $Q\colon \mathcal{D} \to \mathcal{D}_{\mathcal{S}}$ a triangulated functor. Thus, for our purposes, we need a stronger notion. The standard way to approach this is to designate a number of *null objects*, i.e. objects which after localizing should be isomorphic to zero, in such a way that this information is compatible with the distinguished triangles.

Definition 3.47. A full subcategory $\mathcal{N}$ of $\mathcal{D}$ is called a *null system* if

(N0) $\mathcal{N}$ is closed under isomorphism, i.e. if $X \in \mathcal{D}$ is isomorphic to $Y \in \mathcal{N}$, then $X \in \mathcal{N}$,

(N1) $0 \in \mathcal{N}$,

(N2) $\mathcal{N}$ is closed under shifting, i.e. $X \in \mathcal{N}$ if and only if $X[1] \in \mathcal{N}$,

(N3) $\mathcal{N}$ is closed under extensions, i.e. for any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$ where $X, Z \in \mathcal{N}$, then $Y \in \mathcal{N}$.

A (not necessarily full) subcategory satisfying only (N0) is called *replete,* and if said subcategory is full, then one can say it is *strictly full.*

Remark 3.48. The terminology we use here differs from [KS06]. This is due to their terminology generally clashing with many other standards. What we here call *replete* they call *saturated,* but in some resources "saturated" may refer to either: a saturated class of morphisms (related to homotopy theory), or a subcategory which is closed under direct summands.

Remark 3.49. Note that (N3) can actually be strengthened to the following statement: for any distinguished triangle of the form

$$\bullet \longrightarrow \bullet' \longrightarrow \bullet'' \longrightarrow \bullet[1]$$

if any two (non-shifted) objects are in $\mathcal{N}$, then so is the third. In particular, just apply (TR2) along with (N2) and (N3).

Proposition 3.50. *Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{N}$ be a full subcategory of $\mathcal{D}$. Then $\mathcal{N}$ is a null system if and only if $\mathcal{N}$, together with the restriction of $(-)[1]$ to $\mathcal{N}$ and with all distinguished triangles whose objects are in $\mathcal{N}$, is a strictly full triangulated subcategory of $\mathcal{D}$.*

Proof. Let $\mathcal{N}$ be a null system. Then $\mathcal{N}$ is an *additive* strictly full subcategory of $\mathcal{D}$, since by Corollary 3.23 we get that $X \oplus Y \in \mathcal{N}$ for all $X, Y \in \mathcal{N}$. Finally, that $\mathcal{N}$ is triangulated is immediate: it satisfies (TR1)–(TR4) simply because any required distinguished triangle produced in $\mathcal{D}$ using objects of $\mathcal{N}$ ends up in $\mathcal{N}$.

Conversely, suppose $\mathcal{N}$ is a strictly full triangulated subcategory of $\mathcal{D}$. Then (N0), (N1), and (N2) are trivially satisfied. Suppose we have a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{D}$ with $X, Z \in \mathcal{N}$. Then let Y' be a cone of $Z[-1] \to X$ in $\mathcal{N}$, so that we have a morphism of distinguished triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, dashed] & Z \arrow[r] \arrow[d, equal] & X[1] \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1]
\end{tikzcd}$$

by (TR2) and (TR3). Then the dashed arrow must be an isomorphism by Proposition 3.17, so $Y \in \mathcal{N}$. Therefore, $\mathcal{N}$ satisfies (N3), and hence is a null system. ■

We can codify Remark 3.49 as the following:

Corollary 3.51. *Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{N}$ be a null system. Then $\mathcal{N}$ satisfies the following 2-out-of-3 property: for any distinguished triangle of the form*

$$\bullet \longrightarrow \bullet' \longrightarrow \bullet'' \longrightarrow \bullet[1]$$

if any two objects are in $\mathcal{N}$, so is the third.

To any null system, we can associate a reasonable class of morphisms giving a multiplicative system. The idea is as follows: in a triangulated category, we know that a morphism $f\colon X \to Y$ is an isomorphism if and only if there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow X[1].$$

Thus, if we want to make any $Z \in \mathcal{N}$ isomorphic to zero in such a way that we remain compatible with the distinguished triangles, then we ought to invert all maps $f: X \to Y$ for which there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1].$$

Therefore, we make the following definition:

Definition 3.52. Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{N}$ be a null system in $\mathcal{D}$. We then define

$$\mathcal{S}(\mathcal{N}) := \{f\colon X \to Y \mid \text{there exists a d.t. } X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1] \text{ with } Z \in \mathcal{N}\}.$$

Proposition 3.53. *Let $\mathcal{D}$ and $\mathcal{N}$ be as above. Then $\mathcal{S}(\mathcal{N})$ is a multiplicative system.*

Proof. We check that $\mathcal{N}$ is a right multiplicative system, since the left case is dual. If $i\colon X \to Y$ is an isomorphism in $\mathcal{D}$, then we have a distinguished triangle

$$X \xrightarrow{i} Y \longrightarrow 0 \longrightarrow X[1]$$

and since $0 \in \mathcal{N}$ by (N1), we have that $f \in \mathcal{S}(\mathcal{N})$ by definition. Thus, (M1) is satisfied. To check (M2), consider two maps $f: X \to Y$, $g: Y \to Z$, in $\mathcal{S}(\mathcal{N})$. Then (TR4) tells us that we have a distinguished triangle

$$C_f \longrightarrow C_{g\circ f} \longrightarrow C_g \longrightarrow C_f[1]$$

where C_f, $C_{g\circ f}$, and C_g are cones of f, $g \circ f$, and g, respectively. Since $f, g \in \mathcal{S}(\mathcal{N})$, we see that $C_f, C_g \in \mathcal{N}$, and therefore $C_{g\circ f} \in \mathcal{N}$, which finally means that $g \circ f \in \mathcal{S}(\mathcal{N})$.

We now prove (M3). Let $f\colon X \to Y$ be an arbitrary morphism in $\mathcal{D}$, and let $s\colon X \to X'$ be in $\mathcal{S}(\mathcal{N})$. Note that by definition of $\mathcal{S}(\mathcal{N})$, the cone of s is in $\mathcal{N}$. Therefore, (N2) together with (TR2) guarantees that there is some $W \in \mathcal{N}$ and a morphism $h\colon W \to X$ such that

$$W \xrightarrow{h} X \xrightarrow{s} X' \longrightarrow W[1]$$

is a distinguished triangle. Let Y' be a cone of $f \circ h\colon W \to Y$. We then have a morphism of distinguished triangles

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, equal] & X \arrow[r, "s"] \arrow[d, "f"] & X' \arrow[r] \arrow[d, dashed, "g"] & W[1] \arrow[d, equal] \\
W \arrow[r, "f\circ h"] & Y \arrow[r, "t"] & Y' \arrow[r] & W[1]
\end{tikzcd}$$

where the dashed arrow g is obtained using (TR3). This identifies the morphism t as having a cone in $\mathcal{N}$, and therefore $t \in \mathcal{S}(\mathcal{N})$.

Finally, we prove (M4). Here, since we are in an additive category, we may replace f by $f-g$ and g with the zero morphism, so that we need to prove the following: if we have a morphism $f: X \to Y$ and a morphism $s\colon W \to X$ such that $s \in \mathcal{S}(\mathcal{N})$ and $f \circ s = 0$, then there exists a $Z \in \mathcal{D}$ together with a map $t\colon Y \to Z$ such that $t \in \mathcal{S}(\mathcal{N})$ and $t \circ f = 0$. To produce Z and t, we make use of (TR1) and the fact that distinguished triangles give weak kernel/cokernel pairs by Proposition 3.20. In particular, a cone of s is a weak cokernel of s, and so we have a diagram

$$\begin{tikzcd}
W \arrow[r, "s"] & X \arrow[r, "g"] \arrow[dr, "f"] & C_s \arrow[r] \arrow[d, dashed, "h"] & W[1] \\
& & Y &
\end{tikzcd}$$

To obtain Z and t, we complete the map $h\colon C_s \to Y$ to a distinguished triangle

$$C_s \xrightarrow{h} Y \xrightarrow{t} Z \longrightarrow C_s[1]$$

using (TR1). Combining this with the diagram we already had, we get a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "s"] & X \arrow[r, "g"] \arrow[dr, "f"] & C_s \arrow[r] \arrow[d, dashed, "h"] & W[1] \\
& & Y \arrow[d, dashed, "t"] & \\
& & Z &
\end{tikzcd}$$

where the dashed arrows give a distinguished triangle. Since $s \in \mathcal{S}(\mathcal{N})$, we have that $C_s \in \mathcal{N}$, and therefore $t \in \mathcal{S}(\mathcal{N})$. Furthermore, by commutativity we get that

$$t \circ f = t \circ h \circ g = 0 \circ g = 0.$$

This completes the proof. ■

Definition 3.54. Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{N}$ be a null system in $\mathcal{D}$. Then we denote the localization $\mathcal{D}_{\mathcal{S}(\mathcal{N})}$ by $\mathcal{D}/\mathcal{N}$ and call it the *(Verdier) quotient* of $\mathcal{D}$ by $\mathcal{N}$.

Remark 3.55. This notation makes sense, since we are asking that in the quotient all objects of $\mathcal{N}$ become isomorphic to 0.

We will now turn $\mathcal{D}/\mathcal{N}$ into a triangulated category. First, let $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ be the localization functor. Then note that $Q \circ (-)[1] : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ sends morphisms in $\mathcal{S}(\mathcal{N})$ to isomorphisms, so in particular it extends to a functor $(-)[1] : \mathcal{D}/\mathcal{N} \to \mathcal{D}/\mathcal{N}$ sitting in the strictly commutative diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "Q"] \arrow[d, "(-){[1]}"'] \arrow[dr] & \mathcal{D}/\mathcal{N} \arrow[d, dashed, "(-){[1]}"] \\
\mathcal{D} \arrow[r, "Q"] & \mathcal{D}/\mathcal{N}.
\end{tikzcd}$$

Prospectively, this will be the shift functor on $\mathcal{D}/\mathcal{N}$.

Theorem 3.56. *Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{N}$ be a null system in $\mathcal{D}$. Then $\mathcal{D}/\mathcal{N}$ is an additive category. Furthermore, if we equip it with the functor described above and define distinguished triangles to be those that are isomorphic to triangles of the form*

$$Q(X) \xrightarrow{Q(u)} Q(Y) \xrightarrow{Q(v)} Q(Z) \xrightarrow{Q(w)} Q(X[1])$$

where $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is a distinguished triangle in $\mathcal{D}$, then this gives $\mathcal{D}/\mathcal{N}$ the structure of a triangulated category such that the functor $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ is triangulated.

Proof. By Proposition 3.46, the category $\mathcal{D}/\mathcal{N}$ is additive and Q is an additive functor. Given the definitions above, if we can prove that $\mathcal{D}/\mathcal{N}$ is triangulated, then Q is automatically a triangulated functor (on account of the *strict* equality $Q \circ [1] = [1] \circ Q$).

We begin by showing that the shift $(-)[1]$ on $\mathcal{D}/\mathcal{N}$ is an automorphism. For now, to separate it from the shift on $\mathcal{D}$, we will denote it by $(-)[1]_{\mathcal{N}}$. Note that because the diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "Q"] \arrow[dr, "Q"'] & \mathcal{D}/\mathcal{N} \arrow[d, "\mathrm{id}"] \\ & \mathcal{D}/\mathcal{N} \end{tikzcd}$$

commutes, the uniqueness in the definition of the localization implies that the identity on $\mathcal{D}/\mathcal{N}$ is the only functor which satisfies this. Now, let $(-)[-1]_{\mathcal{N}} : \mathcal{D}/\mathcal{N} \to \mathcal{D}/\mathcal{N}$ be defined by the diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "Q"] \arrow[d, "(-)[-1]"'] & \mathcal{D}/\mathcal{N} \arrow[d, dashed] \\ \mathcal{D} \arrow[r, "Q"] & \mathcal{D}/\mathcal{N} \end{tikzcd}$$

and note that

$$[-1]_{\mathcal{N}} \circ [1]_{\mathcal{N}} \circ Q = [-1]_{\mathcal{N}} \circ Q \circ [1] = Q \circ [-1] \circ [1] = Q$$

so that

$$\begin{tikzcd} \mathcal{D} \arrow[r, "Q"] \arrow[dr, "Q"'] & \mathcal{D}/\mathcal{N} \arrow[d, "{[-1]_{\mathcal{N}} \circ [1]_{\mathcal{N}}}"] \\ & \mathcal{D}/\mathcal{N} \end{tikzcd}$$

commutes. By uniqueness, we then have $[-1]_{\mathcal{N}} \circ [1]_{\mathcal{N}} = \mathrm{id}$. An essentially identical computation holds for $[1]_{\mathcal{N}} \circ [-1]_{\mathcal{N}}$, so $[1]_{\mathcal{N}} : \mathcal{D}/\mathcal{N} \to \mathcal{D}/\mathcal{N}$ is an automorphism.

It now remains to check that $\mathcal{D}/\mathcal{N}$ satisfies (TR1)–(TR4). Axiom (TR2) may be deduced by changing all triangles to be of the form

$$Q(X) \xrightarrow{Q(u)} Q(Y) \xrightarrow{Q(v)} Q(Z) \xrightarrow{Q(w)} Q(X[1])$$

and then applying (TR2) to the triangles of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

in $\mathcal{D}$. To check (TR1), consider a morphism $Q(X) \xrightarrow{fs^{-1}} Q(Y)$, i.e. $X \xleftarrow{s} X' \xrightarrow{f} Y$, in $\mathcal{D}/\mathcal{N}$. Then we may take the cone of f in $\mathcal{D}$ to obtain a distinguished triangle $X' \xrightarrow{f} Y \to C_f \to X'[1]$, and we then see that we have an isomorphism of triangles

$$\begin{tikzcd} Q(X') \arrow[r, "Q(f)"] \arrow[d, "Q(s)"', "\wr"] & Q(Y) \arrow[r] \arrow[d, equal] & Q(C_f) \arrow[r] \arrow[d, equal] & Q(X')[1] \arrow[d, "\wr"] \\ Q(X) \arrow[r, "fs^{-1}"] & Q(Y) \arrow[r] & Q(C_f) \arrow[r] & Q(X)[1] \end{tikzcd}$$

where the top triangle is distinguished. Thus, the bottom triangle is also distinguished which verifies (TR1).

To check (TR3), consider a diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & Z_0 \arrow[r] & X_0[1] \arrow[d] \\
X_1 \arrow[r] & Y_1 \arrow[r] & Z_1 \arrow[r] & X_1[1]
\end{tikzcd}$$

and choose isomorphisms

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & Z_i \arrow[r] \arrow[d] & X_i[1] \arrow[d] \\
Q(X_i') \arrow[r] & Q(Y_i') \arrow[r] & Q(Z_i') \arrow[r] & Q(X_i'[1]).
\end{tikzcd}$$

Then we have the diagram

$$\begin{tikzcd}
& Q(X_0') \arrow[rr] \arrow[dd] && Q(Y_0') \arrow[rr] \arrow[dd] && Q(Z_0') \arrow[rr] \arrow[dd, dashed] && Q(X_0'[1]) \arrow[dd] \\
X_0 \arrow[ur] \arrow[rr] \arrow[dd] && Y_0 \arrow[ur] \arrow[rr] \arrow[dd] && Z_0 \arrow[ur] \arrow[rr] \arrow[dd, dashed] && X_0[1] \arrow[ur] \arrow[dd] \\
& Q(X_1') \arrow[rr] && Q(Y_1') \arrow[rr] && Q(Z_1') \arrow[rr] && Q(X_1'[1]) \\
X_1 \arrow[ur] \arrow[rr] && Y_1 \arrow[ur] \arrow[rr] && Z_1 \arrow[ur] \arrow[rr] && X_1[1] \arrow[ur]
\end{tikzcd}$$

where we get the solid vertical arrows in the background by using that the diagonal arrows are isomorphisms, the dashed arrow in the background by applying (TR3) in $\mathcal{D}$, and the dashed arrow in the foreground by again using that the diagonal morphisms are isomorphisms. Axiom (TR4) is proven similarly, i.e. by choosing isomorphic triangles that can be lifted directly to triangles in $\mathcal{D}$, applying (TR4), and then going back by applying $Q(-)$. Thus, we conclude that $\mathcal{D}/\mathcal{N}$ with the given data is a triangulated category. ■

We may restate the universal property of localization with respect to this situation.

Theorem 3.57. *Let $\mathcal{D}$ be a triangulated category, let $\mathcal{N}$ be a null system, and let $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ be the Verdier quotient. Then for all $X \in \mathcal{N}$, we have $Q(X) \cong 0$. Furthermore, if $\mathcal{E}$ is any other triangulated category with a triangulated functor $F\colon \mathcal{D} \to \mathcal{E}$ such that $F(X) \cong 0$ for all $X \in \mathcal{N}$, there is a unique functor $F_{\mathcal{N}}\colon \mathcal{D}/\mathcal{N} \to \mathcal{E}$ such that $F = F_{\mathcal{N}} \circ Q$.*

Proof. For any $X \in \mathcal{N}$, we have a distinguished triangle

$$0 \longrightarrow X \longrightarrow X \longrightarrow 0$$

in $\mathcal{D}$, which shows that $0 \to X$ is in $\mathcal{S}(\mathcal{N})$. Therefore, $0 = Q(0) \cong Q(X)$.

If F sends objects of $\mathcal{N}$ to zero, then it follows that for any morphism $(s\colon X \to Y) \in \mathcal{S}(\mathcal{N})$ we have a distinguished triangle

$$F(X) \xrightarrow{F(s)} F(Y) \longrightarrow 0 \longrightarrow F(X)[1]$$

which implies that $F(s)$ is an isomorphism in $\mathcal{E}$. Therefore, F sends morphisms of $\mathcal{S}(\mathcal{N})$ to isomorphisms, which gives the claimed statement. ■

Corollary 3.58. *Let $\mathcal{D}$ be a triangulated category with a null system $\mathcal{N}$, and consider a cohomological functor $H : \mathcal{D} \to \mathcal{A}$ to an Abelian category $\mathcal{A}$ such that $H(X) = 0$ for all $X \in \mathcal{N}$. Then the induced functor $\mathcal{D}/\mathcal{N} \to \mathcal{A}$ is cohomological.*

Proof. Since every distinguished triangle in $\mathcal{D}/\mathcal{N}$ is (isomorphic to) the image of a distinguished triangle in $\mathcal{D}$, the result follows. ■

The notation for the Verdier quotient can suggest some things which are not quite true. Let us introduce the following:

Definition 3.59. Let $\mathcal{D}$ and $\mathcal{E}$ be triangulated categories, and let $F \colon \mathcal{D} \to \mathcal{E}$ be a triangulated functor. The *kernel* of F is the full subcategory of objects of $\mathcal{D}$ which are sent to zero by F, i.e.

$$\ker F := \{X \in \mathcal{D} \mid F(X) \cong 0\}.$$

The kernel of $F \colon \mathcal{D} \to \mathcal{E}$ is automatically a strictly full triangulated subcategory of $\mathcal{D}$, and hence a null system. One might hope that when $\mathcal{E} = \mathcal{D}/\mathcal{N}$ that we get $\ker Q = \mathcal{N}$, but this is not generally true. In particular, while $\mathcal{N} \subseteq \ker Q$, the kernel may be strictly larger. The problem is this: we know that $\mathcal{N}$ is closed under extension (and hence under direct sums), but we do *not* know that it is closed under taking *summands.* In particular, if $Q(X \oplus Y) = 0$, then it follows that $Q(X) = 0$ and $Q(Y) = 0$ since Q commutes with direct sums. We see that the kernel is a *thick* subcategory.

Definition 3.60. A subcategory $\mathcal{C}$ of a triangulated category $\mathcal{D}$ is *thick* if it is a full triangulated subcategory which is closed under direct summands.

Remark 3.61. In particular, a thick subcategory is closed under shifts, extensions, and direct summands.

We then see that when $\mathcal{N}$ is not thick, the kernel must be larger than $\mathcal{N}$. However, it will turn out that the kernel is not much bigger. In particular, the final aim of this subsection is to prove that $\ker Q = \operatorname{thick}(\mathcal{N})$, i.e. that the kernel of the localization map is the smallest thick subcategory containing $\mathcal{N}$. This fact will also allow us to characterize which morphisms in $\mathcal{D}$ are sent to isomorphisms in $\mathcal{D}/\mathcal{N}$.

We proceed as in [Nee01]. Specifically, the argument contained in this thesis is exactly Lemmas 2.1.31, 2.1.32, 2.1.33, and Proposition 2.1.35 in Neeman's book except with slightly different notation.

Lemma 3.62. *Consider two morphisms $f \colon X \to Y$, $g \colon Y \to Z$ in $\mathcal{D}$. If any two of f, g, $g \circ f$ are in $\mathcal{S}(\mathcal{N})$, then the third is too.*

Proof. If f and g are in $\mathcal{S}(\mathcal{N})$, then trivially $g \circ f \in \mathcal{S}(\mathcal{N})$. In the other cases, we may apply (TR4) to the composite $g \circ f$ to get a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & C_f \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, equal] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, equal] & C_{g\circ f} \arrow[r] \arrow[d, dashed] & X[1] \arrow[d] \\
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & C_g \arrow[r] \arrow[d, equal] & Y[1] \arrow[d] \\
C_f \arrow[r, dashed] & C_{g\circ f} \arrow[r, dashed] & C_g \arrow[r, dashed] & C_f[1]
\end{tikzcd}$$

where we then get the result using the 2-out-of-3 property (N3) of null systems (in particular, see Remark 3.49). ■

Lemma 3.63. *If a morphism*

$$X \xleftarrow{s} X' \xrightarrow{f} X$$

in $\mathcal{D}/\mathcal{N}$ is in the equivalence class of the identity (i.e. if $fs^{-1} = \mathrm{id}_{Q(X)}$), then $f \in \mathcal{S}(\mathcal{N})$.

Proof. Definitionally, if $fs^{-1} = \mathrm{id}_{Q(X)}$ then there exists a $X'' \in \mathcal{D}$, a morphism

$$X \xleftarrow{t} X'' \xrightarrow{h} X,$$

and morphisms $\phi\colon X'' \to X'$, $\psi\colon X'' \to X$ fitting into a commutative diagram

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & X'' \arrow[l, "t"'] \arrow[r, "h"] \arrow[u, "\phi"] \arrow[d, "\psi"'] & X \\
& X \arrow[ul, "\mathrm{id}"] \arrow[ur, "\mathrm{id}_X"'] &
\end{tikzcd}$$

By Lemma 3.62, ϕ and ψ are in $\mathcal{S}(\mathcal{N})$. However, since $\psi = f \circ \phi$, by the same lemma we get that $f \in \mathcal{S}(\mathcal{N})$. ■

Lemma 3.64. *A morphism*

$$X \xleftarrow{s} X' \xrightarrow{g} Y$$

in $\mathcal{D}/\mathcal{N}$ is invertible if and only if there are morphisms f and h such that $g \circ f, h \circ g \in \mathcal{S}(\mathcal{N})$.

Proof. Suppose there exists morphisms f, h such that $g \circ f, h \circ g \in \mathcal{S}(\mathcal{N})$. Then $Q(g \circ f)$ and $Q(h \circ g)$ are invertible, and so we have both a left and right inverse for $Q(g)$. In particular, we have

$$Q(g) \circ Q(f) \circ Q(g \circ f)^{-1} = \mathrm{id}$$

and similarly for the left inverse. Therefore, $Q(g)$ is invertible.

Conversely, suppose $Q(g)$ is invertible. We must produce f and h such that $g \circ f, h \circ g \in \mathcal{S}(\mathcal{N})$. Since $Q(g)$ is invertible, there exists an inverse $Q(g)^{-1} = ft^{-1}\colon Q(Y) \to Q(X')$

$$Y \xleftarrow{t} Y' \xrightarrow{f} X'$$

and we may compose these to get that

$$Y \xleftarrow{t} Y' \xrightarrow{g \circ f} Y$$

is in the equivalence class of the identity. Therefore, $g \circ f \in \mathcal{S}(\mathcal{N})$. The procedure for producing h is dual. ■

With these lemmas in place, we can prove that $\ker Q = \mathrm{thick}(\mathcal{N})$.

Proposition 3.65. *Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{N}$ be a null system. Then the morphism $X \to 0$ in $\mathcal{D}$ becomes an isomorphism in $\mathcal{D}/\mathcal{N}$ if and only if there exists some $Y \in \mathcal{D}$ such that $X \oplus Y \in \mathcal{N}$.*

Proof. Suppose $Q(X) \to 0$ is an isomorphism. By Lemma 3.64, we may choose some $Y \in \mathcal{D}$ such that the composition

$$X \longrightarrow 0 \longrightarrow Y[1]$$

is in $\mathcal{S}(\mathcal{N})$. However, note that we have the distinguished triangle

$$X \xrightarrow{0} Y[1] \longrightarrow (X \oplus Y)[1] \longrightarrow X[1]$$

by Corollary 3.23 and therefore $X \oplus Y \in \mathcal{N}$ by (N2).

Conversely, suppose we have some $Y \in \mathcal{D}$ such that $X \oplus Y \in \mathcal{N}$. We then again have the distinguished triangle

$$X \xrightarrow{0} Y[1] \longrightarrow (X \oplus Y)[1] \longrightarrow X[1]$$

which shows that $0 \colon X \to Y[1]$ is in $\mathcal{S}(\mathcal{N})$. Furthermore, the composition $0 \to X \to 0$ is an isomorphism and hence also in $\mathcal{S}(\mathcal{N})$. Then Lemma 3.64 says that $X \to 0$ is an isomorphism. ■

Corollary 3.66. *Let $\mathcal{D}$ be a triangulated category, let $\mathcal{N}$ be a null system, and let $Q \colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ be the Verdier quotient. Then* $\ker Q = \operatorname{thick}(\mathcal{N})$*. In particular, if $\mathcal{N}$ is furthermore a thick subcategory, then* $\ker Q = \mathcal{N}$*.*

Proof. The null system $\mathcal{N}$ is already a strictly full triangulated subcategory of $\mathcal{D}$, so what remains is to be closed under direct summands. By Proposition 3.65, the kernel $\ker Q$ is exactly the smallest subcategory containing $\mathcal{N}$ which is closed under this operation, and therefore $\ker Q = \operatorname{thick}(\mathcal{N})$. ■

Thus we see that as long as we are working with thick subcategories, the Verdier quotient produces the kinds of results we expect from the notation (and our preconceptions from ordinary abstract algebra). As another corollary to Proposition 3.65, we can give the following (admittedly rather trivial) characterization of morphisms in $\mathcal{D}$ which are sent to isomorphisms in the Verdier quotient $\mathcal{D}/\mathcal{N}$.

Corollary 3.67. *Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{N}$ be a null system. A morphism $f : X \to Y$ in $\mathcal{D}$ is an isomorphism in $\mathcal{D}/\mathcal{N}$ (i.e. $Q(f)$ is an isomorphism) if and only if in every triangle*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

the object Z is a direct summand of an object in $\mathcal{N}$.

Proof. If $Q(f)$ is an isomorphism, then every cone is isomorphic to 0, and hence $Q(Z) \cong 0$. Therefore, Z is a direct summand of an object in $\mathcal{N}$. Conversely, if Z is a direct summand of an object in $\mathcal{N}$ then we have an isomorphism of triangles

$$\begin{tikzcd}
Q(X) \arrow[r, "Q(f)"] \arrow[d, equal] & Q(Y) \arrow[r] \arrow[d, equal] & Q(Z) \arrow[r] \arrow[d] & Q(X)[1] \arrow[d, equal] \\
Q(X) \arrow[r, "Q(f)"] & Q(Y) \arrow[r] & 0 \arrow[r] & Q(X)[1]
\end{tikzcd}$$

so that the bottom triangle is distinguished (since the triangle above is distinguished because Q is triangulated), and hence $Q(f)$ is an isomorphism. ■

3.5 RECOLLEMENT

Definition 3.68. A sequence

$$\mathcal{C} \xrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}$$

of triangulated functors between triangulated categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ is a *Verdier quotient sequence* if P is a fully faithful functor whose essential image is a strictly full thick subcategory of $\mathcal{D}$, and Q identifies $\mathcal{E}$ as the Verdier quotient $\mathcal{D}/P(\mathcal{C})$. We will say it is a *weak* Verdier quotient sequence if the same conditions hold, except $\mathcal{E}$ is only the weak localization rather than strict (see Remark 3.30).

Remark 3.69. The above is totally non-standard terminology, but there also does not appear to be a standard.

Thus, any null system (i.e. strictly full triangulated subcategory) $\mathcal{N}$ of $\mathcal{D}$ gives rise to a Verdier quotient sequence

$$\mathcal{N} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{D}/\mathcal{N}.$$

Indeed, all Verdier quotient sequences are essentially of this form. To make this precise, we begin with the following definition:

Definition 3.70. A *morphism of Verdier quotient sequences* $F\colon (\mathcal{C},\mathcal{D},\mathcal{E},P,Q) \to (\mathcal{C}',\mathcal{D}',\mathcal{E}',P',Q')$ is a triple $F = (F_1, F_2, F_3)$ of functors fitting into a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P"] \arrow[d, "F_1"'] & \mathcal{D} \arrow[r, two heads, "Q"] \arrow[d, "F_2"'] & \mathcal{E} \arrow[d, "F_3"'] \\
\mathcal{C}' \arrow[r, hook, "P'"] & \mathcal{D}' \arrow[r, two heads, "Q'"] & \mathcal{E}'
\end{tikzcd}$$

commutative up to natural isomorphism. We say such a morphism is an isomorphism if the F_i are isomorphisms, and we say it is an equivalence if the F_i are equivalences.

Proposition 3.71. *Consider a Verdier quotient sequence*

$$\mathcal{C} \xhookrightarrow{P'} \mathcal{D} \xrightarrow{Q'} \mathcal{E}.$$

Then there exists a null system $\mathcal{N}$ in $\mathcal{D}$ and an equivalence of Verdier quotient sequences

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P'"] \arrow[d, "\wr"] & \mathcal{D} \arrow[r, two heads, "Q'"] \arrow[d, equal] & \mathcal{E} \arrow[d, "\wr"] \\
\mathcal{N} \arrow[r, hook, "P"] & \mathcal{D} \arrow[r, two heads, "Q"] & \mathcal{D}/\mathcal{N}
\end{tikzcd}$$

where the equivalence $\mathcal{E} \xrightarrow{\sim} \mathcal{D}/\mathcal{N}$ is an isomorphism and the above strictly commutes.

Proof. Let $\mathcal{N}$ be the essential image of $P' : \mathcal{C} \hookrightarrow \mathcal{D}$. Then P' factors through the inclusion $P\colon \mathcal{N} \hookrightarrow \mathcal{D}$ as $P' = P \circ F$ with $F\colon \mathcal{C} \to \mathcal{N}$ being the "codomain restricted" version of P'. Since P' is fully faithful, we know that F is an equivalence $\mathcal{C} \simeq \mathcal{N}$ (in particular, it is trivially fully faithful and essentially surjective). The isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{D}/\mathcal{N}$ follows immediately from the universal property of the (strict) localization. In particular, it depends only on the essential image of P', i.e. on $\mathcal{N}$. ■

In some sense, we therefore see that by considering only Verdier quotient sequences

$$\mathcal{N} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{D}/\mathcal{N}$$

we are essentially seeing all examples. This is in much the same way as any short exact sequence of (not necessarily Abelian) groups

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

can be chosen as isomorphic to

$$1 \longrightarrow H' \longrightarrow G \longrightarrow G/H' \longrightarrow 1$$

and we interpret G as being a "sum" of H and K, though this need of course not be the actual direct sum/product. The point of this interpretation is to say that in such a short exact sequence, G is a combination of H and K. This interpretation also holds for Verdier quotient sequences: in $\mathcal{C} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{E}$, we think of $\mathcal{D}$ as being an extension of $\mathcal{E}$ by $\mathcal{C}$. In the world of Abelian groups (or Abelian categories in general), when such a short exact sequence is split (left or right, these being equivalent there), the middle term is the direct sum of the adjacent terms. If one drops commutativity, the situation is more subtle: for a short exact sequence of arbitrary groups, being left split or being a direct sum behaves the same as in the commutative case, but being right split need not imply being left split nor a direct sum. Instead, the middle term turns into a *semidirect* product instead.

In the world of triangulated categories and Verdier quotient sequences, the situation is a peculiar mixture of the Abelian and arbitrary group cases, as we will see in the comming theorems. In particular, while being left split is the same as being right split (see Theorem 3.87), there is a further distinction between in what way the sequence splits since in this context "split" means "admits an adjoint," and a functor may admit both a left or a right adjoint. The nicest case is where we split in both ways at the same time. This motivates the following definition:

Definition 3.72. A *recollement* is a Verdier quotient sequence together with a number of adjoints

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] & \mathcal{D} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, two heads] & \mathcal{E} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where each arrow is left adjoint to the one below it.

Here, there are a number of interesting statements to make. First, we put no requirements on the adjoints as part of the assumptions, but it turns out that the ones on the left side automatically become essentially surjective and the ones on the right side automatically become fully faithful. Furthermore, one can say quite a lot about the images of the two adjoints on the right. For this, we will first need to know about orthogonal complements.

Definition 3.73. Let $\mathcal{D}$ be an additive category (e.g. triangulated), and let $\mathcal{C}$ be a full subcategory. Define the full subcategories

$$\mathcal{C}^{\perp} := \{Y \in \mathcal{D} \mid \forall X \in \mathcal{C},\, \mathrm{Hom}_{\mathcal{D}}(X,Y) = 0\},$$
$${}^{\perp}\mathcal{C} := \{X \in \mathcal{D} \mid \forall Y \in \mathcal{C},\, \mathrm{Hom}_{\mathcal{D}}(X,Y) = 0\}.$$

Proposition 3.74. *Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{C}$ be a triangulated subcategory of $\mathcal{D}$. Then $\mathcal{C}^{\perp}$ and ${}^{\perp}\mathcal{C}$ are strictly full thick subcategories of $\mathcal{D}$. Furthermore, $\mathcal{C} \cap \mathcal{C}^{\perp} \simeq \mathcal{C} \cap {}^{\perp}\mathcal{C} \simeq \{0\}$.*

Proof. That $\mathcal{C}^{\perp}$ and ${}^{\perp}\mathcal{C}$ are strictly full is obvious. That they are closed under direct summands is also clear: suppose that $X \oplus Y \in \mathcal{C}^{\perp}$, and let $Z \in \mathcal{C}$. Then

$$0 = \mathrm{Hom}_{\mathcal{D}}(X \oplus Y, Z) \cong \mathrm{Hom}_{\mathcal{D}}(X, Z) \oplus \mathrm{Hom}_{\mathcal{D}}(Y, Z) \implies \mathrm{Hom}_{\mathcal{D}}(X, Z) = \mathrm{Hom}_{\mathcal{D}}(Y, Z) = 0.$$

The computation for ${}^{\perp}\mathcal{C}$ is dual. Thus, it only remains to show that $\mathcal{C}^{\perp}$ and ${}^{\perp}\mathcal{C}$ are triangulated subcategories of $\mathcal{D}$. However, this itself is obvious since $\mathcal{C}$ is a triangulated subcategory of $\mathcal{D}$. In particular, if $Z \in \mathcal{C}$ then $Z[i] \in \mathcal{C}$ for all $i \in \mathbb{Z}$, so if $X \in \mathcal{C}^{\perp}$ we have

$$0 = \mathrm{Hom}_{\mathcal{D}}(X, Z) \cong \mathrm{Hom}_{\mathcal{D}}(X[i], Z[i])$$

so that $X[i] \in \mathcal{C}^{\perp}$. Thus $\mathcal{C}^{\perp}$ is closed under shifts, and we now just need it to be closed under extensions. For this, it is enough to show that it is closed under taking cones. Suppose we have a distinguished triangle

$$X' \to X \to Y \to X'[1]$$

with $X, X' \in \mathcal{C}^{\perp}$. Then, for any $Z \in \mathcal{C}$, we have—by the fact that Hom is cohomological—the exact sequence

$$0 = \mathrm{Hom}_{\mathcal{D}}(X'[1]) \to \mathrm{Hom}_{\mathcal{D}}(Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(X, Z) = 0$$

which implies that $\mathrm{Hom}(Y, Z) = 0$. Therefore, $\mathcal{C}^{\perp}$ is thick. The proof for ${}^{\perp}\mathcal{C}$ is identical but dual.

The final statement follows since if $X \in \mathcal{C}$ and $X \in \mathcal{C}^{\perp}$, then for all $Z \in \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{D}}(X, Z) = 0 = \mathrm{Hom}_{\mathcal{D}}(0, Z) \implies X \cong 0$$

by the Yoneda lemma. ■

We can now state the result we want.

Theorem 3.75. *Suppose we have a recollement*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P"] & \mathcal{D} \arrow[l, bend right, "L_P"'] \arrow[l, bend left, "R_P"] \arrow[r, two heads, "Q"] & \mathcal{E}. \arrow[l, bend right, "L_Q"'] \arrow[l, bend left, "R_Q"]
\end{tikzcd}$$

Then

- *(i)* L_P *and* R_P *are essentially surjective,*
- *(ii)* L_Q *and* R_Q *are fully faithful,*
- *(iii)* R_Q *induces an equivalence* $\mathcal{E} \simeq P(\mathcal{C})^{\perp}$ *with quasi-inverse* $P(\mathcal{C})^{\perp} \hookrightarrow \mathcal{D} \xrightarrow{Q} \mathcal{E}$*, and*
- *(iv)* L_Q *induces an equivalence* $\mathcal{E} \simeq {}^{\perp}P(\mathcal{C})$ *with quasi inverse* ${}^{\perp}P(\mathcal{C}) \hookrightarrow \mathcal{D} \xrightarrow{Q} \mathcal{E}$.

We split the proof of this up into several intermediate lemmas.

Lemma 3.76. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be arbitrary categories, suppose we have a fully faithful functor* $F : \mathcal{C} \hookrightarrow \mathcal{D}$*, and suppose it has a left or right adjoint* $G : \mathcal{D} \to \mathcal{C}$*. Then* G *is essentially surjective.*

Proof. Suppose G is a left adjoint of F. Note that we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X,-) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(X),F(-)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(G(F(X)),-)$$

and therefore, by the Yoneda lemma, we have an isomorphism $X \cong G(F(X))$. This means G is essentially surjective: in particular, for every $X \in \mathcal{C}$ there is some $Z \in \mathcal{D}$ (namely, $Z = F(X)$) such that $X \cong G(Z)$. The proof when G is a right adjoint is similar. ■

Lemma 3.77. *Suppose we have arbitrary categories $\mathcal{D}$ and $\mathcal{E}$, and let $F\colon \mathcal{D} \to \mathcal{E}$ be a functor.*

(a) A left adjoint $L : \mathcal{E} \to \mathcal{D}$ with unit $\eta : \mathrm{id}_{\mathcal{E}} \to F \circ L$ and counit $\varepsilon : L \circ F \to \mathrm{id}_{\mathcal{D}}$ is fully faithful if and only if η is an isomorphism. In this case, the natural transformation $F\varepsilon\colon F \circ L \circ F \to F$ is an isomorphism.

(b) Dually, a right adjoint $R\colon \mathcal{E} \to \mathcal{D}$ with unit $\eta'\colon \mathrm{id}_{\mathcal{D}} \to R \circ F$ and counit $\varepsilon'\colon F \circ R \to \mathrm{id}_{\mathcal{E}}$ is fully faithful if and only if ε' is an isomorphism. In this case, the natural transformation $F\eta'\colon F \to F \circ R \circ F$ is an isomorphism.

Proof. We prove only (a) since (b) is totally dual. We have natural transformations

$$\mathrm{Hom}_{\mathcal{E}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(L(X),L(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(X,F(L(Y))).$$

These are $(f\colon X \to Y) \mapsto L(f)$ and $(g\colon L(X) \to L(Y)) \mapsto F(g) \circ \eta_X$, where $\eta\colon \mathrm{id}_{\mathcal{E}} \to F \circ L$ is the unit of the adjunction (L,F). Composing the two, we clearly obtain the map

$$\phi_X\colon \mathrm{Hom}_{\mathcal{E}}(X,Y) \to \mathrm{Hom}_{\mathcal{E}}(X,F(L(Y))), \quad f \mapsto F(L(f)) \circ \eta_X.$$

Since η is a natural transformation, this is the same as the map $(\eta_Y\circ) : \mathrm{Hom}_{\mathcal{E}}(X,Y) \to \mathrm{Hom}_{\mathcal{E}}(X,F(L(Y)))$. In particular, the diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & F(L(X)) \arrow[d, "F(L(f))"] \\
Y \arrow[r, "\eta_Y"] & F(L(Y))
\end{tikzcd}$$

commutes. Furthermore, the maps ϕ_X assemble into a natural transformation $\phi\colon \mathrm{Hom}_{\mathcal{E}}(-,Y) \to \mathrm{Hom}_{\mathcal{E}}(-,F(L(Y)))$. It is then clear that this is a natural isomorphism if and only if L is fully faithful. Explicitly, if L is fully faithful then the first definition of ϕ_X in terms of L and η_X is clearly an isomorphism for any X, and therefore η_Y is an isomorphism. Conversely, if η_Y is an isomorphism then $\phi = (\eta_Y\circ)$ is an isomorphism and hence the map $\mathrm{Hom}_{\mathcal{E}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(L(X),L(Y))$ must be an isomorphism.

Suppose now that L is fully faithful. By the triangle identities, we have the commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\eta F"] \arrow[rr, bend right, "\mathrm{id}_F"'] & F \circ L \circ F \arrow[r, "F\varepsilon"] & F
\end{tikzcd}$$

and since η is a natural isomorphism, we see that ηF is also a natural isomorphism. Therefore, we see that $F\varepsilon$ is a natural isomorphism since it is squeezed between two other isomorphisms. This completes the proof. ■

Lemma 3.78. [Kra22, Prop. 1.1.3] *Suppose we have arbitrary categories $\mathcal{D}, \mathcal{E}$ and an adjoint functor pair $L\colon \mathcal{D} \to \mathcal{E}$, $R\colon \mathcal{E} \to \mathcal{D}$. Let $\mathcal{S}_L$ be those morphisms in $\mathcal{D}$ sent to isomorphisms in $\mathcal{E}$ by L, and let $\mathcal{S}_R$ be those morphisms in $\mathcal{E}$ sent to isomorphisms in $\mathcal{D}$ by R. Then R is fully faithful if and only if the induced functor $L_{\mathcal{S}_L} : \mathcal{D}_{\mathcal{S}_L} \to \mathcal{E}$ is an equivalence, and dually, L is fully faithful if and only if the induced functor $R_{\mathcal{S}_R} : \mathcal{E}_{\mathcal{S}_R} \to \mathcal{D}$ is an equivalence.*

Proof. Since the two assertions of the lemma are dual, we prove only one of them. In particular, we show that R is fully faithful if and only if we have the given equivalence $\mathcal{D}_{\mathcal{S}_L} \simeq \mathcal{E}$. To simplify notation, let $\mathcal{S} = \mathcal{S}_L$.

By Lemma 3.77, it suffices to show that the counit morphisms $\varepsilon_X : L(R(X)) \to X$ being isomorphisms is equivalent to L inducing an equivalence $\mathcal{D}_{\mathcal{S}} \xrightarrow{\sim} \mathcal{E}$. Suppose the ε_X's are isomorphisms, and denote the unit of the adjunction by $\eta\colon \mathrm{id}_{\mathcal{D}} \to R \circ L$. Then $Q \circ R$ is an inverse to the induced functor $L_{\mathcal{S}}\colon \mathcal{D}_{\mathcal{S}} \to \mathcal{E}$. To see this, first note that we have $L_{\mathcal{S}} \circ Q \circ R \cong L \circ R \cong \mathrm{id}_{\mathcal{E}}$ since R is fully faithful. What remains is then the other composition. By Lemma 3.77, the natural transformation $Q\eta\colon Q \to Q \circ R \circ L$ is a natural isomorphism. Thus we compute

$$Q \circ R \circ L_{\mathcal{S}} \circ Q = Q \circ R \circ L \cong Q \cong \mathrm{id}_{\mathcal{D}_{\mathcal{S}}} \circ Q$$

which by the uniqueness in the universal property of the localization implies that $Q \circ R \circ L_{\mathcal{S}} \cong \mathrm{id}_{\mathcal{D}_{\mathcal{S}}}$.

Conversely, suppose that $L_{\mathcal{S}}$ is an equivalence of categories. We aim to show that $\mathrm{id}_{\mathcal{E}}$ is left adjoint to $L \circ R$, since we will then have an isomorphism $\mathrm{Hom}_{\mathcal{E}}(X, Y) \cong \mathrm{Hom}_{\mathcal{E}}(X, L(R(Y)))$ which in particular gives that the counit maps $\varepsilon_Y : L(R(Y)) \to Y$ are isomorphisms. Thus, we show this adjointness claim.

Since $L_{\mathcal{S}}$ is an equivalence, composition with L gives fully faithful functor

$$(\circ L)\colon \mathrm{Fun}(\mathcal{E}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$$

for any category $\mathcal{C}$ since it is the same as the composition

$$\mathrm{Fun}(\mathcal{E}, \mathcal{C}) \xrightarrow[(\circ L_{\mathcal{S}})]{\sim} \mathrm{Fun}(\mathcal{D}_{\mathcal{S}}, \mathcal{C}) \xrightarrow[(\circ Q)]{\sim} \mathrm{Fun}_{\mathcal{S}}(\mathcal{D}, \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{C}).$$

Taking $\mathcal{C} = \mathcal{E}$, we see that there is some $\eta' : \mathrm{id}_{\mathcal{E}} \to L \circ R$ such that $L\eta = \eta' L$. In particular, we take the inverse image of $L\eta$ under the isomorphism $\mathrm{Hom}(\mathrm{id}_{\mathcal{E}}, L \circ R) \xrightarrow{\sim} \mathrm{Hom}(L, L \circ R \circ L)$. Prospectively, η' is the unit of our desired adjunction (while ε will be the counit). By the triangle identities for η, ε we have that

$$\mathrm{id}_L = \varepsilon L \circ L\eta = \varepsilon L \circ \eta' L = (\varepsilon \circ \eta')L.$$

Therefore $\varepsilon \circ \eta' = \mathrm{id}_{\mathrm{id}_{\mathcal{E}}}$. This gives one of the triangle identities. The other follows by applying L to the other triangle identity for η, ε as follows:

$$\mathrm{id}_R = R\varepsilon \circ \eta R \implies \mathrm{id}_{L \circ R} = (L \circ R)\varepsilon \circ L\eta R = (L \circ R)\varepsilon \circ \eta'(L \circ R).$$

This proves that $\mathrm{id}_{\mathcal{E}}$ is left adjoint to $L \circ R$, which finishes the proof. ■

Lemma 3.79. [MM92, p. 369, Lemma VII.4.1] *Let $\mathcal{D}$ and $\mathcal{E}$ be arbitrary categories. Suppose we have a functor $F\colon \mathcal{D} \to \mathcal{E}$ together with a left adjoint $L\colon \mathcal{E} \to \mathcal{D}$ and a right adjoint $R\colon \mathcal{E} \to \mathcal{D}$. Then L is fully faithful if and only if R is fully faithful.*

Proof. Let $\eta\colon \mathrm{id}_{\mathcal{E}} \to F \circ L$ be the unit of the adjunction (L, F), and let $\varepsilon' : F \circ R \to \mathrm{id}_{\mathcal{E}}$ be the counit of the adjunction (F, R). For all $X, Y \in \mathcal{E}$, we then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(X), R(Y)) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathcal{E}}(X, F(R(Y))) \arrow[d, "(\varepsilon'_Y \circ)"] \\
\mathrm{Hom}_{\mathcal{E}}(F(L(X)), Y) \arrow[r, "(\circ \eta_X)"] & \mathrm{Hom}_{\mathcal{E}}(X, Y)
\end{tikzcd}$$

where, for clarity, we note that the isomorphisms are given explicitly by

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}}(L(X), R(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(X, F(R(Y))), && f \mapsto F(f) \circ \eta_X, \\
&\mathrm{Hom}_{\mathcal{D}}(L(X), R(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(F(L(X)), Y), && f \mapsto \varepsilon'_Y \circ F(f).
\end{aligned}$$

so the commutativity of the diagram is entirely trivial. This implies that all the η_X's are isomorphisms if and only if all the ε'_Y's are isomorphisms. Therefore, η is an isomorphism if and only if ε' is an isomorphism, which by Lemma 3.77 implies that L is fully faithful if and only if R is fully faithful. ■

With these lemmas in place, we can put together a proof of the theorem of interest.

Proof of Theorem 3.75. (i) is exactly Lemma 3.76.

(ii) R_Q satisfies the conditions of Lemma 3.78, and is thus fully faithful. Thus, by Lemma 3.79 we automatically get that L_Q is fully faithful.

(iii) We check that the essential image of R_Q is $P(\mathcal{C})^{\perp}$. Suppose we have some object of $P(\mathcal{C})$. It is then isomorphic to an object of the form $P(X)$, where $X \in \mathcal{C}$. Then, for any $Y \in \mathcal{E}$, we have

$$\mathrm{Hom}_{\mathcal{D}}(P(X), R_Q(Y)) \cong \mathrm{Hom}_{\mathcal{E}}(Q(P(X)), Y) \cong \mathrm{Hom}_{\mathcal{E}}(0, Y) = 0.$$

Therefore, $R_Q(\mathcal{E}) \subseteq P(\mathcal{C})^{\perp}$. Conversely, let $Y \in P(\mathcal{C})^{\perp}$. We have the unit map $\eta_Y : Y \to R_Q(Q(Y))$, and by Lemma 3.77 we know that $Q(\eta_Y)$ is an isomorphism. Therefore, there is some $Z \in \mathcal{C}$ and a distinguished triangle

$$Y \xrightarrow{\eta_Y} R_Q(Q(Y)) \longrightarrow P(Z) \longrightarrow Y[1]$$

afterwhich we observe that both Y and $R_Q(Q(Y))$ are in $P(\mathcal{C})^{\perp}$ and therefore, since $P(\mathcal{C})$ is a full triangulated subcategory of $\mathcal{D}$ and hence $P(\mathcal{C})^{\perp}$ is thick, we have $P(Z) \in P(\mathcal{C}) \cap P(\mathcal{C})^{\perp}$. Therefore, $P(Z) = 0$, so η_Y is an isomorphism $Y \cong R_Q(Q(Y))$. Hence, $R_Q(\mathcal{E}) = P(\mathcal{C})^{\perp}$.

(iv) The proof of this is entirely dual to (iii). In particular, the inclusion $L_Q(\mathcal{E}) \subseteq {}^{\perp}P(\mathcal{C})$ is obvious, and the other inclusion comes from observing that, by Lemma 3.77, the counit map $\varepsilon_X \colon L_Q(Q(X)) \to X$ becomes an isomorphism after applying Q. Thus, for $X \in {}^{\perp}P(\mathcal{C})$, one gets a distinguished triangle

$$L_Q(Q(X)) \longrightarrow X \longrightarrow P(Z) \longrightarrow L_Q(Q(X))[1]$$

which implies $P(Z) \in P(\mathcal{C}) \cap {}^{\perp}P(\mathcal{C}) \simeq \{0\}$ so ε_X is an isomorphism. ■

Remark 3.80. As a result of the above, we see that all recollements are actually of the form

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] & \mathcal{D} \arrow[r, two heads] \arrow[l, two heads, bend right] \arrow[l, two heads, bend left] & \mathcal{E}. \arrow[l, hook, bend right] \arrow[l, hook, bend left]
\end{tikzcd}$$

Proposition 3.81. *Consider a recollement*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P"] & \mathcal{D} \arrow[l, bend right, two heads, "L_P"'] \arrow[l, bend left, two heads, "R_P"] \arrow[r, two heads, "Q"] & \mathcal{E}. \arrow[l, bend right, hook] \arrow[l, bend left, hook]
\end{tikzcd}$$

Then $\ker L_P = {}^{\perp}P(\mathcal{C})$ *and* $\ker R_P = P(\mathcal{C})^{\perp}$.

Proof. We show one equality since the other is dual. Suppose that $X \in \mathcal{D}$. Then

$$\begin{aligned}
L_P(X) = 0 &\iff \forall Y' \in \mathcal{C},\, \mathrm{Hom}_{\mathcal{C}}(L_P(X), Y') = 0 \\
&\iff \forall Y' \in \mathcal{C},\, \mathrm{Hom}_{\mathcal{D}}(X, P(Y')) = 0 \\
&\iff X \in {}^{\perp}P(\mathcal{C}).
\end{aligned}$$

This completes the proof. ■

Together with Lemma 3.78, we can use this proposition to say some interesting things. We first introduce some terminology for convenience.

Definition 3.82. A Verdier quotient sequence

$$\mathcal{C} \xhookrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}$$

is *reflective* if P and Q both admit a left adjoint, and it is *coreflective* if P and Q both admit a right adjoint.

Remark 3.83. Note that a recollement is then a Verdier quotient sequence which is both reflective and coreflective.

Remark 3.84. The terminology we use here is based on the terminologies *(co)reflective subcategory* and *(co)reflective localization.* This is not a universal choice: for example, in [Kra22], what we call a coreflective Verdier sequence he calls a *localization sequence.*

Proposition 3.85. *Suppose we have a reflective Verdier quotient sequence*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P"] & \mathcal{D} \arrow[l, bend right, two heads, "L_P"'] \arrow[r, two heads, "Q"] & \mathcal{E}. \arrow[l, bend right, hook, "L_Q"']
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook, "L_Q"] & \mathcal{D} \arrow[l, bend left, two heads, "Q"] \arrow[r, two heads, "L_P"] & \mathcal{C} \arrow[l, bend left, hook, "P"]
\end{tikzcd}$$

is a coreflective weak Verdier quotient sequence. The dual statement also holds. That is, if we have a coreflective Verdier quotient sequence, then this induces a reflective weak Verdier quotient sequence.

Proof. This is essentially a consequence of Lemma 3.78. Certainly, the essential image of L_Q is a strictly full thick subcategory of $\mathcal{D}$, so what remains is really only to check that $\mathcal{C}$ is the appropriate Verdier quotient. Note that for a morphism $f : X \to Y$ in $\mathcal{D}$, $L_P(f)$ is an isomorphism if and only if

$$L_P(X) \longrightarrow L_P(Y) \longrightarrow 0 \longrightarrow L_P(X)[1]$$

is a distinguished triangle, and this is true if and only if in every distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

we have $Z \in \ker L_P = {}^{\perp}P(\mathcal{C}) = L_Q(\mathcal{E})$. Thus, since P is fully faithful we see that, by Lemma 3.78, we have a canonical equivalence $\mathcal{D}/L_Q(\mathcal{E}) \simeq \mathcal{C}$. ■

Remark 3.86. There is no version of Lemma 3.78 which replaces equivalence with isomorphism, and so there is no way to prove a version of the above proposition where we drop the "weak" part of the weak Verdier quotient sequences.

The above proposition in particular implies that from a recollement

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P" description] & \mathcal{D} \arrow[l, two heads, bend right, "L_P"'] \arrow[l, two heads, bend left, "R_P"] \arrow[r, two heads, "Q" description] & \mathcal{E} \arrow[l, hook, bend right, "L_Q"'] \arrow[l, hook, bend left, "R_Q"]
\end{tikzcd}$$

we can extract (at least "weak") coreflective and reflective Verdier localization sequences

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook, "L_Q"] & \mathcal{D} \arrow[l, two heads, bend left, "Q"] \arrow[r, two heads, "L_P"] & \mathcal{C} \arrow[l, hook, bend left, "P"] & \text{and} & \mathcal{E} \arrow[r, hook, "R_Q"'] & \mathcal{D} \arrow[l, two heads, bend right, "Q"'] \arrow[r, two heads, "R_P"'] & \mathcal{C}. \arrow[l, hook, bend right, "P"']
\end{tikzcd}$$

More can be said about recollements. One item of interest is that there are superfluous assumptions in the definition: if we are to trust our intuition that Verdier quotient sequences $\mathcal{C} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{E}$ are like short exact sequences of Abelian groups, then one should recognize that a short exact sequence splits on the right precisely when it splits on the left. The analogue of this statement turns out to also be true for Verdier quotient sequences.

Theorem 3.87. *Suppose we have a Verdier quotient sequence*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P"] & \mathcal{D} \arrow[r, two heads, "Q"] & \mathcal{E}.
\end{tikzcd}$$

Then we have the following two collections of equivalent statements:

(i) The following are equivalent:

  (a) P has a right adjoint R_P,

  (b) Q has a right adjoint R_Q, and

  (c) for all $X \in \mathcal{D}$ there are $X' \in P(\mathcal{C})$, $X'' \in P(\mathcal{C})^{\perp}$ and a distinguished triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1].$$

(ii) The following are equivalent:

  (a) P has a left adjoint L_P,

  (b) Q has a left adjoint L_Q,

  (c) for all $X \in \mathcal{D}$ there are $X' \in {}^{\perp}P(\mathcal{C})$, $X'' \in P(\mathcal{C})$ and a distinguished triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1].$$

In the case of (i), the equivalent conditions induce an equivalence $\mathcal{D}/P(\mathcal{C})^{\perp} \xrightarrow{\sim} \mathcal{C}$ and an equality ${}^{\perp}(P(\mathcal{C})^{\perp}) = P(\mathcal{C})$. In the case of (ii), the equivalent conditions induce an equivalence $\mathcal{D}/{}^{\perp}P(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$ and an equality $({}^{\perp}P(\mathcal{C}))^{\perp} = P(\mathcal{C})$.

Proof. The idea is that the distinguished triangles tell us what the adjoints are. Furthermore, (i) and (ii) are totally dual to each other, so it suffices to prove only one of them. We will prove (ii).

Suppose we have (ii)(c). We will construct the functors L_P and L_Q. To construct L_P, first let $X'' \cong P(X_0'')$. Letting $W \cong P(W_0) \in P(\mathcal{C})$, we have natural morphisms in exact rows

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{D}}(X'[1], W) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathcal{D}}(X'', W) \arrow[r] \arrow[d, "\wr"] & \operatorname{Hom}_{\mathcal{D}}(X, W) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathcal{D}}(X', W) \arrow[d, equal] \\
0 \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(X_0'', W_0) \arrow[r, "\sim"] & \operatorname{Hom}_{\mathcal{D}}(X, W) \arrow[r] & 0
\end{tikzcd}$$

so we have a natural isomorphism $\operatorname{Hom}_{\mathcal{C}}(X_0'', -) \cong \operatorname{Hom}_{\mathcal{D}}(X, P(-))$. Therefore, the assignment $X \mapsto X_0''$ defines the desired left adjoint L_P. In particular, a map $X \to Y$ composes to a map $X \to Y''$, which under the aforementioned natural isomorphism defines a unique map $L_P(X) \to L_P(Y)$. This proves that (ii)(c) implies (ii)(a). Constructing L_Q is similar. In particular, letting $W \in {}^{\perp}P(\mathcal{C})$ this time, we see by applying $\operatorname{Hom}(W, -)$ that we have the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{{}^{\perp}P(\mathcal{C})}(W, X') \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(W, X) \longrightarrow 0$$

so that we have a natural isomorphism $\operatorname{Hom}_{{}^{\perp}P(\mathcal{C})}(-, X') \cong \operatorname{Hom}_{\mathcal{D}}(-, X)$. Thus we have a *right* adjoint R to the inclusion $\iota \colon {}^{\perp}P(\mathcal{C}) \hookrightarrow \mathcal{D}$. Observe that for all $Z_0 \in \mathcal{C}$, we have $R(P(Z_0)) = 0$ since

$$\operatorname{Hom}_{{}^{\perp}P(\mathcal{C})}(-, R(P(Z_0))) \cong \operatorname{Hom}_{\mathcal{D}}(\iota(-), P(Z_0)) = 0.$$

Therefore, the functor R factors through Q and we obtain a functor $F \colon \mathcal{E} \to {}^{\perp}P(\mathcal{C})$ such that $F \circ Q = R$. In fact, this functor is an equivalence of categories by Lemma 3.78 since the left adjoint of R, namely the inclusion ι, is fully faithful. The idea is now to show that $\iota \circ F$ is left adjoint to Q. To do this, note that we have natural isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{E}}(-, Q(-)) &\cong \operatorname{Hom}_{{}^{\perp}P(\mathcal{C})}(F(-), (F \circ Q)(-)) \\
&\cong \operatorname{Hom}_{{}^{\perp}P(\mathcal{C})}(F(-), R(-)) \cong \operatorname{Hom}_{\mathcal{D}}((\iota \circ F)(-), -)
\end{aligned}$$

which finishes the proof of (ii)(c) $\implies$ (ii)(b).

For the last bit, we need to show both that (ii)(a) implies (ii)(c) and that (ii)(b) implies (ii)(c). The idea here is again quite simple, and the approach is to take the cone/cocone of the appropriate unit/counit. Thus, assume (ii)(a). We can then consider the unit $\eta \colon \mathrm{id}_{\mathcal{D}} \to P \circ L_P$, and in particular the component $\eta_X \colon X \to P(L_P(X))$ for any X. Taking the cocone, we obtain a distinguished triangle

$$X' \longrightarrow X \longrightarrow P(L_P(X)) \longrightarrow X'[1].$$

Since P is fully faithful, we know by Lemma 3.77 that the natural transformation $L_P \to L_P \circ P \circ L_P$ is a natural isomorphism, so by applying L_P we have a distinguished triangle

$$L_P(X') \longrightarrow L_P(X) \xrightarrow{\sim} L_P(X) \longrightarrow L_P(X')[1]$$

which implies that $L_P(X') = 0$. Therefore, for any $Z \in \mathcal{C}$, we have

$$\operatorname{Hom}_{\mathcal{D}}(X', P(Z)) \cong \operatorname{Hom}_{\mathcal{E}}(L_P(X'), Z) = \operatorname{Hom}_{\mathcal{E}}(0, Z) = 0.$$

This completes the proof of (ii)(a) $\implies$ (ii)(c).

Finally, assume (ii)(b). Then we have the counit $\varepsilon' : L_Q \circ Q \to \mathrm{id}_{\mathcal{D}}$ and in particular the component $\varepsilon'_X : L_Q(Q(X)) \to X$. Taking the cone, we have a distinguished triangle

$$L_Q(Q(X)) \longrightarrow X \longrightarrow X'' \longrightarrow L_Q(Q(X))[1]$$

and we know by Lemma 3.77 that $Q \circ L_Q \circ Q \to Q$ is a natural isomorphism. Therefore, applying Q gives the distinguished triangle

$$Q(X) \xrightarrow{\sim} Q(X) \longrightarrow Q(X'') \longrightarrow Q(X)[1]$$

so that $Q(X'') = 0$, hence $Q(X) \in \ker Q = P(\mathcal{C})$. This shows (ii)(b) $\implies$ (ii)(c), and thus we have completed all the equivalences.

That we have an equivalence $\mathcal{D}/{}^{\perp}P(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$ is now an immediate consequence of Lemma 3.78. In particular, it is exactly Proposition 3.85.

We now show $({}^{\perp}P(\mathcal{C}))^{\perp} = P(\mathcal{C})$ when (ii) is fullfilled. The inclusion $P(\mathcal{C}) \subseteq ({}^{\perp}P(\mathcal{C}))^{\perp}$ is obvious. Conversely, let $Y \in ({}^{\perp}P(\mathcal{C}))^{\perp}$. Then, by the constructions in the proof that (ii)(c) implies (ii)(a) and (ii)(b), we have a distinguished triangle

$$L_Q(Q(Y)) \longrightarrow Y \longrightarrow P(L_P(Y)) \longrightarrow L_Q(Q(Y))[1].$$

We then note that $L_Q(Q(Y)) \in L_Q(\mathcal{E}) = {}^{\perp}P(\mathcal{C})$, so in particular $\mathrm{Hom}_{\mathcal{D}}(L_Q(Q(Y)), Y) = 0$. Thus, after shifting, we have a distinguished triangle

$$Y \longrightarrow P(L_P(Y)) \longrightarrow L_Q(Q(Y))[1] \xrightarrow{0} Y[1]$$

which exhibits $P(L_P(Y))$ as the sum of Y and $L_Q(Q(Y))[1]$ by Corollary 3.23. In particular, Y is a direct summand of $P(L_P(Y))$ which is in the thick subcategory $P(\mathcal{C})$, hence $Y \in P(\mathcal{C})$. Therefore, $P(\mathcal{C}) = ({}^{\perp}P(\mathcal{C}))^{\perp}$. ■

Remark 3.88. For an alternative proof that $({}^{\perp}P(\mathcal{C}))^{\perp} = P(\mathcal{C}) = {}^{\perp}(P(\mathcal{C})^{\perp})$, see the computations in Example 5.49.

Corollary 3.89. *Suppose we have a recollement*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P"] & \mathcal{D} \arrow[l, two heads, bend right, "L_P"'] \arrow[l, two heads, bend left, "R_P"] \arrow[r, two heads, "Q"] & \mathcal{E}. \arrow[l, hook, bend right, "L_Q"'] \arrow[l, hook, bend left, "R_Q"]
\end{tikzcd}$$

Then, for all $X \in \mathcal{D}$, there are distinguished triangles

$$L_Q(Q(X)) \xrightarrow{\varepsilon''_X} X \xrightarrow{\eta_X} P(L_P(X)) \longrightarrow L_Q(Q(X))[1]$$

$$P(R_P(X)) \xrightarrow{\varepsilon'_X} X \xrightarrow{\eta'''_X} R_Q(Q(X)) \longrightarrow P(R_P(X))[1]$$

where η_X is the unit of the adjunction (L_P, P), ε''_X is the counit of the adjunction (L_Q, Q), η''_X is the unit of the adjunction (Q, R_Q), and ε'_X is the counit of the adjunction (P, R_P).

Proof. This is simply by the constructions of the relevant adjoints in the proof that (ii)(c) $\implies$ (ii)(a) and (ii)(b) in Theorem 3.87. In particular, this gives the first distinguished triangle. The second one is just a result of the dual argument. ■

3.6 NOTES ON ENHANCEMENTS

As pointed out in Sections 3.1 and 3.2, triangulated categories have the problem that the operation of taking cones is not in general functorial, in particular as a consequence of the non-canonicity of the morphism in (TR3). This is a fundamental, unavoidable problem with triangulated categories. To see this, consider any triangulated category $\mathcal{D}$ and an object $X \in \mathcal{D}$. We then have the distinguished triangle

$$X \longrightarrow X \longrightarrow 0 \longrightarrow X[1]$$

which we can use to produce an explicit example of non-canonicity in (TR3). In particular, we note that by shifting it appropriately in two different ways, we get the two following morphisms of distinguished triangles:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X[1] \arrow[r, "\mathrm{id}"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "0"] & X[1] \arrow[d] \\
0 \arrow[r] & X[1] \arrow[r, "\mathrm{id}"] & X[1] \arrow[r] & 0
\end{tikzcd}$$

As a result, it is necessarily impossible for there to be a unique choice of filling in (TR3), and we can conclude that taking cones cannot be a "universal construction" in the context of triangulated categories.

The above technical difficulty is also inherent in the natural examples we come across: while the cone construction yields a functor $\mathrm{Mor}(\mathbf{C}(\mathcal{A})) \to \mathbf{C}(\mathcal{A})$, it fails to yield a functor $\mathrm{Mor}(\mathbf{K}(\mathcal{A})) \to \mathbf{K}(\mathcal{A})$. The reason for this is simple: suppose we have morphisms $f\colon X^\bullet \to Y^\bullet$ and $g\colon Z^\bullet \to W^\bullet$ of chain complexes, and we have a morphism $(u,v)\colon f \to g$ in $\mathrm{Mor}(\mathbf{K}(\mathcal{A}))$, it is not too hard to check that the equations induced by having a functorial cone forces this to lift to a morphism in $\mathrm{Mor}(\mathbf{C}(\mathcal{A}))$. In particular, if the cone operation in $\mathbf{K}(\mathcal{A})$ was functorial, then we would have $v \circ f = g \circ u$ *strictly,* i.e. not up to homotopy. This is a useful point to keep in mind, because it indicates that what we actually *want* as a kind of universal property would be that if we *supply* the homotopy $v \circ f \Rightarrow g \circ u$, then we should have a uniquely determined induced map on the cones of f and g.

The fact that cones do not behave as well as one might want is one of the central technical difficulties with triangulated categories. Indeed, the cone fails to be given by a universal property due in large part to the failure of triangulated categories (e.g. $\mathbf{K}(\mathcal{A})$) to explicitly remember homotopy data. Rectifying this requires some form of modification to actual category theory which lets one escape strict non-canonicity and remember some (coherent) homotopy data. Specifically, one needs something like $(\infty,1)$-categories.

Various attempts at such structures have been made, including such things as Grothendieck's *derivators* (possibly the best one can do in a 1-categorical world), but the approach which is most popular at present uses simplicial models for $(\infty,1)$-categories, namely quasicategories for which [Lur09] is a standard encyclopedic reference. Henceforth, when we say "∞-category," we mean in the sense of Lurie. Using an ∞-categorical framework, one can note that cones are canonical up to some level of *(coherent) homotopy* (giving them a genuine universal property), and thus ∞-categories (settings for doing homotopy coherent mathematics) can be of use in this situation.

With ∞-categories at hand, one can define *stable* ∞-categories, essentially ∞-categorical (or homotopy) versions of Abelian categories, where we recall that triangulated categories are supposed to be a kind of homotopy version of Abelian categories also. Any ∞-category has an associated 1-category called the *homotopy category,* and stable ∞-categories have the property that their homotopy category automatically has a canonical triangulated structure; in other words, stable ∞-categories are an ∞-categorical "refinement" or "enhancement" of triangulated categories. A discussion of this can be found in [Lur17]. An appealing aesthetic feature here is that stable ∞-categories are defined by properties rather than by having a certain structure endowed on them.

Stable ∞-categories also enjoy a number of other nice properties that triangulated categories do not. For example, applying some reasonable categorical construction to a triangulated category does not usually yield a triangulated category. However, in the world of stable ∞-categories, this is no longer the case and one can, for example, take limits or (filtered) colimits of them to obtain new stable ∞-categories.

One way in which this improved behavior of stable ∞-categories exhibits itself is in the theory of recollements. In the world of triangulated categories, when recollements do exist, they behave reasonably well. However, they do not always exist. Specifically, suppose we have two triangulated categories $\mathcal{C}$ and $\mathcal{E}$, and a triangulated functor $F\colon \mathcal{E} \to \mathcal{C}$ along which we wish to "glue" $\mathcal{C}$ and $\mathcal{E}$. Sometimes we get lucky and this exists (in which case the gluing functor F is given by $R_P \circ L_Q$, in the notation of Section 3.5), but in general this is simply not the case. The construction one would like to go through with does not work. On the other hand, if we replace $\mathcal{C}$ and $\mathcal{E}$ with stable ∞-categories, and F with an exact functor of such, then it is always possible to construct a recollement (with F as the gluing functor). For a resource on this topic, see [Lur17, Appendix A.8], and in particular Remark A.8.12 there. For a perhaps less imposing resource, see [DJW19, §1] or [Sha22, §2].

Remark 3.90. While stable ∞-categories are an analogue of Abelian categories in the context of ∞-categories, it should be noted that they are not an "enlargement" of the category of Abelian categories. In particular, given an Abelian category $\mathcal{A}$, one may reasonably ask if (the nerve of) $\mathcal{A}$ is a stable ∞-category, and this is never the case unless $\mathcal{A}$ is (equivalent to) the zero category. Heuristically, the reason for this is pretty simple: the fact that stable ∞-categories come equipped with a canonical suspension (i.e. shift) operation given by pushing out along two zero maps implies that for any $X \in \mathcal{A}$, we would necessarily have an ∞-Cartesian square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X[1]
\end{tikzcd}$$

but since this would have to be a Cartesian square in $\mathcal{A}$ as a 1-category, this would imply that $X = 0$, and so $\mathcal{A} \simeq \{0\}$.

Remark 3.91. On the other hand, there are non-trivial triangulated categories which are also Abelian categories.

4 Derived Categories

Derived categories form natural categories within which to formulate (co)homological theorems. Essentially, the idea is to recognize that, for the purposes of homological algebra, we may always replace a chain complex by a chain complex which has identical (co)homological properties. Thus, it makes sense to consider a category which only cares about chain complexes “up to cohomology.” This is exactly the derived category.

Mostly, this section derives from the exposition of derived categories in [KS06].

4.1 Chain Homotopies & Quasi-Isomorphisms

Let $\mathcal{A}$ be an Abelian category. Recall that at the end of Section 2, we defined the cohomology functor $\mathrm{H}^i\colon \mathbf{C}(\mathcal{A}) \to \mathcal{A}$. In particular, we saw that a morphism $f\colon A^\bullet \to B^\bullet$ induces morphisms $\mathrm{H}^i(f) : \mathrm{H}^i(A^\bullet) \to \mathrm{H}^i(B^\bullet)$ for each $i \in \mathbb{Z}$. This assignment is not injective: it can occur that the induced morphisms on cohomology agree without the original morphisms being the same. A way to produce examples of this is to consider *chain homotopies.*

Definition 4.1. Let $f, g\colon A^\bullet \to B^\bullet$ be morphisms of chain complexes in $\mathcal{A}$. A *chain homotopy* $\eta\colon f \Rightarrow g$ is a collection of morphisms $\eta^i\colon A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^i - g^i = \eta^{i+1} \circ d_A^i + d_B^{i-1} \circ \eta^i.$$

Pictorially, we have the (non-commutative) diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[dl] \arrow[d, shift right] \arrow[d, shift left] & A^i \arrow[r, "d_A^i"] \arrow[dl, "\eta^i"'] \arrow[d, shift right, "g^i"'] \arrow[d, shift left, "f^i"] & A^{i+1} \arrow[r] \arrow[dl, "\eta^{i+1}"] \arrow[d, shift right] \arrow[d, shift left] & \cdots \arrow[dl] \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"'] & B^i \arrow[r, "d_B^i"'] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

We define a relation $\sim_\mathrm{h}$ on $\mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(A^\bullet, B^\bullet)$ by $f \sim_\mathrm{h} g$ if there exists a chain homotopy $f \Rightarrow g$. We say f is *nullhomotopic* if $f \sim_\mathrm{h} 0$.

Proposition 4.2. *The relation* $\sim_\mathrm{h}$ *on* $\mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(A^\bullet, B^\bullet)$ *is an equivalence relation. Furthermore, it is compatible with the addition of morphisms in the sense that if* $f \sim_\mathrm{h} h$ *and* $g \sim_\mathrm{h} k$*, then* $(f + g) \sim_\mathrm{h} (h + k)$*, and additionally it is compatible with composition in the sense that if we have maps*

$$\begin{tikzcd}
C'^\bullet \arrow[r, "h"] & A^\bullet \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B^\bullet \arrow[r, "k"] & C^\bullet
\end{tikzcd}$$

such that $f \sim_\mathrm{h} g$*, then* $f \circ h \sim g \circ h$ *and* $k \circ f \sim k \circ g$*. In other words, the system of nullhomotopic morphisms forms a two-sided ideal in* $\mathbf{C}(\mathcal{A})$*. In particular,* $\{f : A^\bullet \to B^\bullet \mid f \sim_\mathrm{h} 0\}$ *forms a subgroup of the morphisms* $A^\bullet \to B^\bullet$*.*

Proof. Clearly, $f \sim_\mathrm{h} f$ using the zero maps. If $f \sim_\mathrm{h} g$ via a chain homotopy $\eta : f \Rightarrow g$, then taking the maps $-\eta^i$ gives a chain homotopy $-\eta\colon g \Rightarrow f$. In particular,

$$g^i - f^i = -(f^i - g^i) = -(\eta^{i+1} \circ d_A^i + d_B^{i-1} \circ \eta^i) = (-\eta^{i+1}) \circ d_A^i + d_B^{i-1} \circ (-\eta^i).$$

Finally, if we have chain homotopies $\eta\colon f \Rightarrow g$ and $\sigma : g \Rightarrow h$, then $\eta - \sigma$ is a chain homotopy $f \Rightarrow h$. In particular,

$$\begin{aligned} f^i - h^i = (f^i - g^i) - (h^i - g^i) &= \eta^{i+1} \circ d^i + d^{i-1} \circ \eta^i - \sigma^{i+1} \circ d^i - d^{i-1} \circ \sigma^i \\ &= (\eta^{i+1} - \sigma^{i+1}) \circ d^i + d^{i-1} \circ (\eta^i - \sigma^i). \end{aligned}$$

Therefore, $f \sim_{\mathrm{h}} g$ and $g \sim_{\mathrm{h}} h \implies f \sim_{\mathrm{h}} h$, so $\sim_{\mathrm{h}}$ is an equivalence relation.

Suppose that $f \sim_{\mathrm{h}} h$ and $g \sim_{\mathrm{h}} k$. It is then clear that $(f+g) \sim_{\mathrm{h}} (h+k)$ since $f+g-h-k = (f-h)+(g-k)$. Similarly, since $h \circ f - h \circ g = h \circ (f-g)$ we have

$$h^i \circ (f^i - g^i) = (h^i \circ \eta^{i+1}) \circ d^i + h^i \circ d^{i-1} \circ \eta^i = (h^i \circ \eta^{i+1}) \circ d^i + d^{i-1} \circ (h^{i-1} \circ \eta^i)$$

so we obtain a chain homotopy $h \circ f \Rightarrow h \circ g$. The other case is identical. ■

Corollary 4.3. *The following data assembles into a category* $\mathbf{K}(\mathcal{A})$*, called the* homotopy category of chain complexes *in* $\mathcal{A}$*: objects are simply objects of* $\mathbf{C}(\mathcal{A})$*. The morphisms are defined by* $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, B^\bullet) := \operatorname{Hom}_{\mathbf{C}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim_{\mathrm{h}}$.

The remarkable thing about this particular equivalence relation is that it is compatible with cohomology. One should expect this from the terminology chosen: in topology, two homotopic continuous maps induce the same map on cohomology. In the context of homological algebra, we have the

Proposition 4.4. *Let* $f, g : A^\bullet \to B^\bullet$ *be morphisms of chain complexes, and suppose that* $f \sim_{\mathrm{h}} g$*. Then* $\mathrm{H}^i(f) = \mathrm{H}^i(g)$ *for all* $i \in \mathbb{Z}$*. In particular, let* $U\colon \mathbf{C}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ *be the obvious functor. Then* $\mathrm{H}^i : \mathbf{C}(\mathcal{A}) \to \mathcal{A}$ *lifts to a functor* $\mathbf{K}(\mathcal{A}) \to \mathcal{A}$*, also denoted by* H^i*, such that* $\mathrm{H}^i \circ U = \mathrm{H}^i$*.*

Proof. Let $f\colon A^\bullet \to B^\bullet$ be a morphism of chain complexes. It suffices to show that if $f \sim_{\mathrm{h}} 0$, then $\mathrm{H}^i(f) = 0$ for all $i \in \mathbb{Z}$. We have a chain homotopy $\eta\colon f \Rightarrow 0$, i.e. a collection of maps $\eta^i\colon A^i \to B^{i-1}$ in the (non-commutative) diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, phantom] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"] \arrow[dl] & A^i \arrow[r, "d^i"] \arrow[d, "f^i"] \arrow[dl, "\eta^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] \arrow[dl, "\eta^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d^{i-1}"'] & B^i \arrow[r, "d^i"'] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $f^i = \eta^{i+1} \circ d^i + d^{i-1} \circ \eta^i$. Recall the diagram

$$\begin{tikzcd}
A^{i-1} \arrow[r, two heads] \arrow[ddd, "f^{i-1}"'] \arrow[drr, "d_A^{i-1}"'] & \operatorname{im} d_A^{i-1} \arrow[rr, hook] \arrow[dr, hook] \arrow[ddd] & & \ker d_A^i \arrow[r, two heads] \arrow[dl, hook'] \arrow[ddd] & \mathrm{H}^i(A^\bullet) \arrow[ddd, "\mathrm{H}^i(f)"] \\
& & A^i \arrow[ddd, "f^i"] & & \\
& & & & \\
B^{i-1} \arrow[r, two heads] \arrow[drr, "d_B^{i-1}"'] & \operatorname{im} d_B^{i-1} \arrow[rr, hook] \arrow[dr, hook] & & \ker d_B^i \arrow[r, two heads] \arrow[dl, hook'] & \mathrm{H}^i(B^\bullet) \\
& & B^i & &
\end{tikzcd}$$

To show that $\mathrm{H}^i(f) = 0$, it suffices to show that the composition $\ker d^i_A \to \ker d^i_B \twoheadrightarrow \mathrm{H}^i(B^\bullet)$ is zero. Specifically, showing this implies that one may choose $\mathrm{H}^i(f) = 0$ to make the right-most square commute, and then uniqueness gives the result.

Observe that, since the composition $\ker d^i \hookrightarrow A^i \xrightarrow{d^i_A} A^{i+1}$ is zero by definition, the composition of this with η^{i+1} is zero. In particular, the composition

$$\ker d^i \hookrightarrow A^i \xrightarrow{\eta^{i+1}\circ d^i} B^i$$

is zero. Furthermore, the map $d^{i-1}_B \circ \eta^i$ factorizes through $\operatorname{im} d^{i-1}_B$ as

$$\begin{tikzcd}
A^i \arrow[r, "\eta^i"] & B^{i-1} \arrow[rr, "d^{i-1}_B"] \arrow[dr, two heads] & & B^i \\
& & \operatorname{im} d^{i-1}_B \arrow[ur, hook] &
\end{tikzcd}$$

and therefore the composition of the morphisms $\ker d^i_A \to \ker d^i_B \twoheadrightarrow \mathrm{H}^i(B^\bullet)$ is equal to

$$\ker d^i_A \hookrightarrow A^i \xrightarrow{\eta^i} B^{i-1} \twoheadrightarrow \operatorname{im} d^{i-1}_B \hookrightarrow \ker d^i_B \twoheadrightarrow \mathrm{H}^i(B^\bullet)$$

which is zero by definition. Therefore $\mathrm{H}^i(f) = 0$, as desired. ■

Thus, as far as cohomology is concerned, the category $\mathbf{K}(\mathcal{A})$ of chain complexes up to homotopy is just as good as the category $\mathbf{C}(\mathcal{A})$. We will now give a first "naive" definition of the derived category.

Definition 4.5. Let $f\colon A^\bullet \to B^\bullet$ be a morphism of chain complexes in an Abelian category $\mathcal{A}$. We say f is a *quasi-isomorphism* if $\mathrm{H}^i(f)$ is an isomorphism for every $i \in \mathbb{Z}$. We then say that $A^\bullet$ and $B^\bullet$ are *quasi-isomorphic.*

The derived category should be a category which only cares about complexes "up to cohomology." Thus, it makes sense to define the derived category of $\mathcal{A}$ as the category of chain complexes $\mathbf{C}(\mathcal{A})$ localized at the quasi-isomorphisms. However, as observed in the earlier discussion of localization, this gives us relatively little control over the behavior of the localization, and in fact we seem to lose knowledge of what properties of $\mathcal{A}$ (and $\mathbf{C}(\mathcal{A})$) transfer over to the derived category. Therefore, we would like to impose additional structure that allows us to better track these kinds of things. In particular, we want the derived category to be a triangulated category.

To obtain the triangulated structure, we cannot proceed using $\mathbf{C}(\mathcal{A})$. Instead, we impose a triangulated structure upon the homotopy category of chain complexes $\mathbf{K}(\mathcal{A})$, after which we take a Verdier quotient by those chain complexes quasi-isomorphic to zero (which turns out to be the same as localizing at quasi-isomorphisms). Note that the notion of a quasi-isomorphism is independent of homotopy by Proposition 4.4.

4.2 THE TRIANGULATED STRUCTURE ON $\mathbf{K}(\mathcal{A})$

The first step in giving $\mathbf{K}(\mathcal{A})$ a triangulated structure is to satisfy (TR1), i.e. we need to produce a cone C_f for any morphism $f : X^\bullet \to Y^\bullet$ of chain complexes. We do this following [KS06], though in that book the authors perform the construction more generally for an additive category with translation (of which $\mathbf{C}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$ are examples).

Definition 4.6. Let $\mathcal{A}$ be an Abelian category, and let $X^\bullet \in \mathbf{C}(\mathcal{A})$ be a chain complex. Define $X[1]^\bullet$ to be the complex given by $X[1]^i = X^{i+1}$ and $d^i_{X[1]} = -d^{i+1}_X$. This data forms an additive automorphism $(-)[1] \colon \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{A})$ which sends a morphism $f : X^\bullet \to Y^\bullet$ to the morphism $f[1]$ defined by $f[1]^i = f^{i+1}$. We write $(-)[i]$ for $(-)[1]$ applied i times.

Remark 4.7. Note that $d_{X[1]} = -d_X[1]$, and that we previously defined a shift functor $(-)(1)$ which, on morphisms, also acts by $f(1)^i = f^{i+1}$. Thus $X[1]^\bullet$ is the chain complex $(X(1)^\bullet, -d_X(1))$. Furthermore, note that d_X defines a morphism of chain complexes $X^\bullet \to X[1]^\bullet$.

Definition 4.8. Let $f \colon X^\bullet \to Y^\bullet$ be a morphism of chain complexes in $\mathcal{A}$. The *mapping cone* of f is the complex $C^\bullet_f$ defined by $C^i_f := X^{i+1} \oplus Y^i$, i.e. $C^\bullet_f = X(1)^\bullet \oplus Y^\bullet$, and with differential given by the matrix

$$d_{C_f} := \begin{pmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{pmatrix} = \begin{pmatrix} -d_X[1] & 0 \\ f[1] & d_Y \end{pmatrix}.$$

Remark 4.9. Hence, the differential on C_f is defined, by universal property, by the maps $d_{X[1]}$ and 0 defining a map $X(1) \oplus Y \to X(2)$, the maps $f[1]$ and d_Y defining a map $X(1) \oplus Y \to Y(1)$, and thus this data combining together to a map $X(1) \oplus Y \to X(2) \oplus Y(1)$. For clarity, it fits into a diagram

$$\begin{tikzcd}
 & X(1)^\bullet \arrow[r, "d_{X[1]}"] \arrow[d] \arrow[ddr, "f{[1]}"] & X(2)^\bullet & \\
C^\bullet_f \arrow[r, equal] & X(1)^\bullet \oplus Y^\bullet \arrow[r, dashed] & X(2)^\bullet \oplus Y(1)^\bullet \arrow[u] \arrow[d] \arrow[r, equal] & C_f(1)^\bullet \\
 & Y^\bullet \arrow[r, "d_Y"'] \arrow[u] \arrow[uur, "0"'] & Y(1)^\bullet &
\end{tikzcd}$$

where the dashed arrow is then the differential $C^\bullet_f \to C_f(1)^\bullet$.

Proposition 4.10. *Consider the category* $\operatorname{Mor}(\mathbf{C}(\mathcal{A}))$ *of morphisms in* $\mathbf{C}(\mathcal{A})$*. Then the operation* $f \mapsto C^\bullet_f$ *defines a functor* $\operatorname{Mor}(\mathbf{C}(\mathcal{A})) \to \mathbf{C}(\mathcal{A})$*, which is defined on morphisms* (f, g) : $(X^\bullet \to Y^\bullet) \to (X'^\bullet \to Y'^\bullet)$ *in* $\operatorname{Mor}(\mathbf{C}(\mathcal{A}))$ *by*

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] \arrow[d, "u"] & X'^\bullet \arrow[d, "v"] \\
Y^\bullet \arrow[r, "g"'] & Y'^\bullet
\end{tikzcd} \mapsto f[1] \oplus g \colon C^\bullet_u \to C^\bullet_v$$

Proof. As long as $f[1] \oplus g$ actually defines a morphism of chain complexes as desired, this will automatically define a functor. Thus, we need only check that $f[1] \oplus g$ is a morphism $C^\bullet_u \to C^\bullet_v$. This is a computation:

$$\begin{aligned}
d^i_{C_v} \circ (f^{i+1} \oplus g^i) &= \begin{pmatrix} -d^{i+1}_{X'} & 0 \\ v^{i+1} & d^i_{Y'} \end{pmatrix} \begin{pmatrix} f^{i+1} & 0 \\ 0 & g^i \end{pmatrix} \\
&= \begin{pmatrix} -d^{i+1}_{X'} \circ f^{i+1} & 0 \\ v^{i+1} \circ f^{i+1} & d^i_{Y'} \circ g^i \end{pmatrix} \\
&= \begin{pmatrix} -f^{i+2} \circ d^i_X & 0 \\ g^{i+1} \circ v^{i+1} & g^{i+1} \circ d^i_{Y'} \end{pmatrix} \\
&= \begin{pmatrix} f^{i+2} & 0 \\ 0 & g^{i+1} \end{pmatrix} \begin{pmatrix} -d^{i+1}_X & 0 \\ u^{i+1} & d^i_Y \end{pmatrix} = (f^{i+2} \oplus g^{i+1}) \circ d^i_{C_u}
\end{aligned}$$

as desired. ■

Proposition 4.11. *Let $f: X^\bullet \to Y^\bullet$ be a morphism of chain complexes in $\mathcal{A}$. Then $C_f^\bullet$ is a chain complex which fits into a triangle*

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{\alpha_f} C_f^\bullet \xrightarrow{\beta_f} X[1]^\bullet$$

where α_f is defined by the maps $0\colon Y^\bullet \to X[1]^\bullet$ and id_Y and β_f is defined by the maps $\mathrm{id}_{X[1]}$ and $0\colon Y^\bullet \to X[1]^\bullet$.

Proof. To see that $d_{C_f}[1] \circ d_{C_f} = 0$, just note that

$$\begin{pmatrix} -d_X[2] & 0 \\ f[2] & d_Y[1] \end{pmatrix} \circ \begin{pmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{pmatrix} = \begin{pmatrix} d_X[2] \circ d_X[1] & 0 \\ f[2] \circ d_{X[1]} + d_Y[1] \circ f[1] & d_Y[1] \circ d_Y \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ f[2] \circ d_{X[1]} + d_Y[1] \circ f[1] & 0 \end{pmatrix}.$$

One then notes that, since f is a morphism of chain complexes, we have

$$f[2] \circ d_{X[1]} + d_Y[1] \circ f[1] = (-f[1] \circ d_X + d_Y \circ f)[1] = 0$$

and therefore $(C_f^\bullet, d_{C_f})$ is a chain complex. ■

Remark 4.12. This proposition shows us an obvious obstruction to making $\mathbf{C}(\mathcal{A})$ a triangulated category (or at least in such a way that the above triangles become distinguished). In particular, Lemma 3.14 says that the composition $\alpha_f \circ f$ should be zero. This is trivially not true! Indeed, we can write α_f in coordinates as the map $(0, \mathrm{id}_Y)$, and $\alpha_f \circ f$ as $(0, f)$ which is clearly non-zero when f is non-zero. Thus, the given triangle cannot be distinguished in any triangulated structure on $\mathbf{C}(\mathcal{A})$.

Lemma 4.13. *Let $\mathcal{A}$ be an Abelian category. Then the shift functor $(-)[1]$ preserves homotopy equivalence. In particular, if $f, g\colon X^\bullet \to Y^\bullet$ are morphisms of chain complexes such that $f \sim_{\mathrm{h}} g$, then $f[1] \sim_{\mathrm{h}} g[1]$. Thus, $(-)[1]$ induces an automorphism $\mathbf{K}(\mathcal{A}) \xrightarrow{\sim} \mathbf{K}(\mathcal{A})$.*

Proof. It suffices to assume $g = 0$, so that $f \sim_{\mathrm{h}} 0$. Thus we have a homotopy $\eta\colon f \Rightarrow 0$, i.e. an equation

$$f^i = d_Y^{i-1} \circ \eta^i + \eta^{i+1} \circ d_X^i.$$

Incrementing i by one, we get

$$f^{i+1} = d_Y^i \circ \eta^{i+1} + \eta^{i+2} \circ d_X^{i+1}$$

so that

$$f[1]^i = d_{Y[1]}^{i-1} \circ (-\eta^{i+1}) + (-\eta^{i+1}) \circ d_{X[1]}^i$$

providing the desired homotopy. ■

Theorem 4.14. *Let $\mathcal{A}$ be an Abelian category. Then $\mathbf{K}(\mathcal{A})$ is a triangulated category when endowed with the shift functor $(-)[1] : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ and the class of distinguished triangles given by all triangles isomorphic to those of the form*

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow C_f^\bullet \longrightarrow X[1]^\bullet.$$

Proof. The second half of (TR1) is obvious by definition of the distinguished triangles. The first half (i.e. that $X^\bullet \to X^\bullet \to 0 \to X[1]^\bullet$ is distinguished) will follow from (TR2) and the fact that the mapping cone of the zero map $0 \to X^\bullet$ is $X^\bullet$ itself, and fits in the triangle

$$0 \longrightarrow X^\bullet \xrightarrow{\text{id}} X^\bullet \longrightarrow 0[1] = 0.$$

Thus, it remains to prove (TR2), (TR3) and (TR4). Consider a distinguished triangle

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X[1]^\bullet$$

and note that we may assume that $Z^\bullet = C_f^\bullet$, $u = f$, $v = \alpha_f$, and $w = \beta_f$ since the above triangle is necessarily isomorphic to that. We construct a map $\phi\colon X[1]^\bullet \to C_{\alpha_f}^\bullet$ which is an isomorphism in $\mathbf{K}(\mathcal{A})$ and for which the diagram

$$\begin{tikzcd} Y^\bullet \arrow[r, "\alpha_f"] \arrow[d, equal] & C_f^\bullet \arrow[r, "\beta_f"] \arrow[d, equal] & X[1]^\bullet \arrow[r, "-f[1]"] \arrow[d, "\phi"] & Y[1]^\bullet \arrow[d, equal] \\ Y^\bullet \arrow[r, "\alpha_f"] & C_f^\bullet \arrow[r, "\alpha_{\alpha_f}"] & C_{\alpha_f}^\bullet \arrow[r, "\beta_{\alpha_f}"] & Y[1]^\bullet \end{tikzcd} \tag{3}$$

commutes. We will first summarize the items at play. Note that, forgetting the differential structure,

$$C_f^\bullet = X[1]^\bullet \oplus Y^\bullet, \quad C_{\alpha_f}^\bullet = Y[1]^\bullet \oplus X[1]^\bullet \oplus Y^\bullet.$$

These have differentials

$$d_{C_f} = \begin{pmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{pmatrix}, \quad d_{C_{\alpha_f}} = \begin{pmatrix} d_{Y[1]} & 0 & 0 \\ 0 & d_{X[1]} & 0 \\ \text{id}_{Y[1]} & f[1] & d_Y \end{pmatrix} = \begin{pmatrix} -d_Y[1] & 0 & 0 \\ 0 & -d_X[1] & 0 \\ \text{id}_{Y[1]} & f[1] & d_Y \end{pmatrix}.$$

On coordinates, we have

$$\alpha_f = \begin{pmatrix} 0 \\ \text{id}_Y \end{pmatrix}, \qquad \alpha_{\alpha_f} = \begin{pmatrix} 0 & 0 \\ \text{id}_{X[1]} & 0 \\ 0 & \text{id}_Y \end{pmatrix},$$

$$\beta_f = \begin{pmatrix} \text{id}_{X[1]} & 0 \end{pmatrix}, \qquad \beta_{\alpha_f} = \begin{pmatrix} \text{id}_{Y[1]} & 0 & 0 \end{pmatrix}.$$

We now define ϕ and its homotopy inverse ψ. Let ϕ be the map given on coordinates by $(-f[1], \text{id}_{X[1]}, 0)$, and let $\psi : C_{\alpha_f}^\bullet \to X[1]^\bullet$ be given by the matrix $\begin{pmatrix} 0 & \text{id}_{X[1]} & 0 \end{pmatrix}$. These are compatible with the differentials, and so define actual morphisms of chain complexes.

Next, observe that trivially $\psi \circ \phi = \text{id}_{X[1]}$. We expect this to be the homotopy inverse of ϕ, so we want $\phi \circ \psi \sim_{\text{h}} \text{id}$. One then computes that $\phi \circ \psi$ is given by the matrix

$$\phi \circ \psi = \begin{pmatrix} -f[1] \\ \text{id}_{X[1]} \\ 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \text{id}_{X[1]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -f[1] & 0 \\ 0 & \text{id}_{X[1]} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We define a homotopy $\text{id} \Rightarrow \psi \circ \psi$ by the matrix

$$\eta = \begin{pmatrix} 0 & 0 & \text{id}_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is indeed a homotopy of the given type: observe that

$$
\begin{aligned}
&\eta^{i+1} \circ d^i_{C_{\alpha_f}} + d^{i-1}_{C_{\alpha_f}} \circ \eta^i \\
&= \begin{pmatrix} 0 & 0 & \mathrm{id}_{Y^{i+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} -d_Y^{i+1} & 0 & 0 \\ 0 & -d_X^{i+1} & 0 \\ \mathrm{id}_{Y^{i+1}} & f^{i+1} & d_Y^i \end{pmatrix} + \begin{pmatrix} -d_Y^i & 0 & 0 \\ 0 & -d_X^i & 0 \\ \mathrm{id}_{Y^i} & f^i & d_Y^{i-1} \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & \mathrm{id}_{Y^i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_{Y^{i+1}} & f^{i+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{Y^i} \end{pmatrix} = \mathrm{id}_{C_{\alpha_f}} - \phi^i \circ \psi^i.
\end{aligned}
$$

This proves that ψ is a homotopy inverse to ϕ. That the right-most square in (3) commutes (in $\mathbf{C}(\mathcal{A})$) is easily verified by a matrix computation. The middle square only commutes in $\mathbf{K}(\mathcal{A})$, which one verifies by checking that $\psi \circ \alpha_{\alpha_f} = \beta_f$; again, this is an easy matrix computation. This proves that the given morphism of triangles is an isomorphism in $\mathbf{K}(\mathcal{A})$, so the triangle

$$Y^\bullet \xrightarrow{\alpha_f} C_f^\bullet \xrightarrow{\beta_f} X[1]^\bullet \xrightarrow{-f} Y[1]^\bullet$$

is distinguished, which proves (TR2). Observe that by previous remarks, this also proves the full (TR1).

To prove (TR3), consider first the diagram

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] \arrow[d, "u"] & Y^\bullet \arrow[r, "\alpha_f"] \arrow[d, "v"] & C_f^\bullet \arrow[r, "\beta_f"] & X[1]^\bullet \arrow[d, "{u[1]}"] \\
Z^\bullet \arrow[r, "g"] & W^\bullet \arrow[r, "\alpha_g"] & C_g^\bullet \arrow[r, "\beta_g"] & Z[1]^\bullet
\end{tikzcd}$$

which we are to fill. This commutes only in $\mathbf{K}(\mathcal{A})$, so we are given some homotopy

$$\sigma \colon v \circ f \Rightarrow g \circ u.$$

We then build a morphism

$$w \colon C_f^\bullet \to C_g^\bullet, \quad w := \begin{pmatrix} u[1] & 0 \\ \sigma[1] & v \end{pmatrix}, \text{ i.e. } w^i = \begin{pmatrix} u^{i+1} & 0 \\ \sigma^{i+1} & v^i \end{pmatrix}.$$

This is compatible with the differentials. In particular,

$$
\begin{aligned}
d^{i-1} \circ w^{i-1} = \begin{pmatrix} -d_Z^i & 0 \\ g^i & d_W^{i-1} \end{pmatrix} \circ \begin{pmatrix} u^i & 0 \\ \sigma^i & v^{i-1} \end{pmatrix} &= \begin{pmatrix} -d_Z^i \circ u^i & 0 \\ g^i \circ u^i + d_W^{i-1} \circ \sigma^i & d_W^{i-1} \circ v^{i-1} \end{pmatrix} \\
&= \begin{pmatrix} -u^{i+1} \circ d_X^i & 0 \\ v^i \circ f^i - \sigma^{i+1} \circ d_X^i & v^i \circ d_Y^{i-1} \end{pmatrix} \\
&= \begin{pmatrix} u^{i+1} & 0 \\ \sigma^{i+1} & v^i \end{pmatrix} \circ \begin{pmatrix} -d_X^i & 0 \\ f^i & d_Y^{i-1} \end{pmatrix} = w^i \circ d^{i-1}.
\end{aligned}
$$

Therefore, w defines an actual morphism of chain complexes. We then have $w \circ \alpha_f = \alpha_g \circ v$ and $u[1] \circ \beta_f = \beta_g \circ w$ by easy matrix computations. This proves (TR3).

Finally, we prove (TR4). Suppose we have morphisms $X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Y'^\bullet$. We wish to fill in the dashed arrows in the diagram

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] \arrow[d, equal] & Y^\bullet \arrow[r, "\alpha_f"] \arrow[d, "g"] & C_f^\bullet \arrow[r, "\beta_f"] \arrow[d, dashed] & X[1]^\bullet \arrow[d, equal] \\
X^\bullet \arrow[r, "g\circ f"] \arrow[d, "f"'] & Y'^\bullet \arrow[r, "\alpha_{g\circ f}"] \arrow[d, equal] & C_{g\circ f}^\bullet \arrow[r, "\beta_{g\circ f}"] \arrow[d, dashed] & X[1]^\bullet \arrow[d, "f[1]"] \\
Y^\bullet \arrow[r, "g"] \arrow[d, "\alpha_f"'] & Y'^\bullet \arrow[r, "\alpha_g"] \arrow[d, "\alpha_{g\circ f}"'] & C_g^\bullet \arrow[r, "\beta_g"] \arrow[d, equal] & Y[1]^\bullet \arrow[d, "\alpha_f[1]"] \\
C_f^\bullet \arrow[r, dashed] & C_{g\circ f}^\bullet \arrow[r, dashed] & C_g^\bullet \arrow[r, dashed] & C_f[1]^\bullet
\end{tikzcd}$$

which we note commutes in $\mathbf{C}(\mathcal{A})$ (if we ignore the dashed arrows). The fact that this commutes in $\mathbf{C}(\mathcal{A})$, and not just in $\mathbf{K}(\mathcal{A})$, means that in the construction from (TR3) we may choose the trivial homotopy for each of the squares on the left to obtain maps

$$C_f^\bullet \xrightarrow[u]{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & g \end{pmatrix}} C_{g\circ f}^\bullet \xrightarrow[v]{\begin{pmatrix} f[1] & 0 \\ 0 & \mathrm{id} \end{pmatrix}} C_g^\bullet \xrightarrow[w]{\begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{pmatrix}} C_f[1]^\bullet.$$

It is clear that these choices make the diagram above commute. Thus, what remains is to show that it is distinguished. We do this using the same method we used to prove (TR2). Thus consider the mapping cone of u and note that, forgetting the differential, we have

$$C_u = X[2]^\bullet \oplus Y[1]^\bullet \oplus X[1]^\bullet \oplus Y'^\bullet.$$

We define the maps $\phi\colon C_u^\bullet \to C_g^\bullet$ and $\psi\colon C_g^\bullet \to C_u^\bullet$ by the matrices

$$\phi = \begin{pmatrix} 0 & \mathrm{id}_{Y[1]} & f[1] & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Y'} \end{pmatrix} \qquad \psi = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y[1]} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{Y'} \end{pmatrix}.$$

It can be checked that these are compatible with the differentials. Clearly, we then have $\phi \circ \psi = \mathrm{id}_{C_g}$. We have a homotopy $\sigma\colon 1 \Rightarrow \psi \circ \phi$ given by the matrix

$$\sigma = \begin{pmatrix} 0 & 0 & \mathrm{id}_{X[1]} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One may explicitly check that this is a homotopy of the given type just as was done in the proof of (TR2). We will omit this since it is uninteresting (particularly in light of the fact that we have already done a similar computation). In any case, we see that ψ is an inverse of ϕ in $\mathbf{K}(\mathcal{A})$. It is easily observed that the diagram

$$\begin{tikzcd}
C_f^\bullet \arrow[r, "u"] \arrow[d, equal] & C_{g\circ f}^\bullet \arrow[r, "\alpha_u"] \arrow[d, equal] & C_u^\bullet \arrow[r, "\beta_u"] \arrow[d, shift right, "\phi"'] & C_f[1]^\bullet \arrow[d, equal] \\
C_f^\bullet \arrow[r, "u"] & C_{g\circ f}^\bullet \arrow[r, "v"] & C_g^\bullet \arrow[r, "w"] \arrow[u, shift right, "\psi"'] & C_f[1]^\bullet
\end{tikzcd}$$

commutes in $\mathbf{K}(\mathcal{A})$ (in particular, $\phi \circ \alpha_u = v$ and $\beta_u \circ \psi = w$) giving that the lower triangle is distinguished. This proves (TR4), and so we have completed the proof that $\mathbf{K}(\mathcal{A})$ is a triangulated category when endowed with the given structure. ■

Proposition 4.15. [KS06, Thm. 12.2.4 & Cor. 12.2.5] *Let $\mathcal{A}$ be an Abelian category, and consider an exact sequence*

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

in $\mathcal{A}$. Then, for all $i \in \mathbb{Z}$,

(i) the induced sequence

$$\mathrm{H}^i(X^\bullet) \to \mathrm{H}^i(Y^\bullet) \to \mathrm{H}^i(Z^\bullet)$$

is exact,

(ii) there exists a $\delta^i \colon \mathrm{H}^i(Z^\bullet) \to \mathrm{H}^{i+1}(X^\bullet)$, functorial in the exact sequence, such that

$$\mathrm{H}^i(Y^\bullet) \to \mathrm{H}^i(Z^\bullet) \overset{\delta^i}{\to} \mathrm{H}^{i+1}(X^\bullet) \to \mathrm{H}^{i+1}(Y^\bullet)$$

is exact, and

(iii) the functor $\mathrm{H}^i \colon \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ is cohomological.

Proof. For (i) and (ii), first note that for all $i \in \mathbb{Z}$ we have the commutative diagram (with exact rows)

$$\begin{tikzcd}
& \operatorname{coker} d_X^{i-1} \arrow[r] \arrow[d, "u"] & \operatorname{coker} d_Y^{i-1} \arrow[r, two heads] \arrow[d, "v"] & \operatorname{coker} d_Z^{i-1} \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & \ker d_X^{i+1} \arrow[r, hook] & \ker d_Y^{i+1} \arrow[r] & \ker d_Z^{i+1} &
\end{tikzcd} \tag{4}$$

as a result of the following very large commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[dd] & 0 \arrow[d] \\
X^{i-1} \arrow[r] \arrow[dd, hook] & X^i \arrow[rr] \arrow[dd, hook] \arrow[dr, two heads] & & X^{i+1} \arrow[rr] \arrow[dd, hook] & & X^{i+2} \arrow[dd, hook] \\
& & \operatorname{coker} d_X^{i-1} \arrow[ur] \arrow[rr, dashed] \arrow[dd] & & \ker d_X^{i+1} \arrow[ul, hook] \arrow[dd, hook] & \\
Y^{i-1} \arrow[r] \arrow[dd, two heads] & Y^i \arrow[rr] \arrow[dd, two heads] \arrow[dr, two heads] & & Y^{i+1} \arrow[rr] \arrow[dd, two heads] & & Y^{i+2} \arrow[dd, two heads] \\
& & \operatorname{coker} d_Y^{i-1} \arrow[ur] \arrow[rr, dashed] \arrow[dd, two heads] & & \ker d_Y^{i+1} \arrow[ul, hook] \arrow[dd] & \\
Z^{i-1} \arrow[r] \arrow[dd] & Z^i \arrow[rr] \arrow[dd] \arrow[dr, two heads] & & Z^{i+1} \arrow[rr] \arrow[dd] & & Z^{i+2} \arrow[dd] \\
& & \operatorname{coker} d_Z^{i-1} \arrow[ur] \arrow[rr, dashed] \arrow[d] & & \ker d_Z^{i+1} \arrow[ul, hook] & \\
0 & 0 & 0 & 0 & & 0
\end{tikzcd}$$

which is obtained by repeatedly using universal properties. One observes that there are natural isomorphisms

$$\mathrm{H}^i(X^\bullet) \cong \ker(\operatorname{coker} d_X^{i-1} \to X^{i+1}) \cong \ker(\operatorname{coker} d_X^{i-1} \to \ker d_X^{i+1})$$
$$\mathrm{H}^{i+1}(X^\bullet) \cong \operatorname{coker}(X^i \to \ker d^i) \cong \operatorname{coker}(\operatorname{coker} d_X^{i-1} \to \ker d_X^i)$$

since the map $\ker d_X^{i+1} \hookrightarrow X^{i+1}$ is a monomorphism and the map $X^i \twoheadrightarrow \operatorname{coker} d_X^{i-1}$ is an epimorphism. Applying the snake lemma (see, for example, [KS06, p. 297, Lemma 12.1.1]) to (4) then gives us the long exact sequence

$$\begin{tikzcd}
\ker u \arrow[r] \arrow[d, "\wr"] & \ker v \arrow[r] \arrow[d, "\wr"] & \ker w \arrow[r] \arrow[d, "\wr"] & \operatorname{coker} u \arrow[r] \arrow[d, "\wr"] & \operatorname{coker} v \arrow[r] \arrow[d, "\wr"] & \operatorname{coker} w \arrow[d, "\wr"] \\
\mathrm{H}^i(X^\bullet) \arrow[r, dashed] & \mathrm{H}^i(Y^\bullet) \arrow[r, dashed] & \mathrm{H}^i(Z^\bullet) \arrow[r, dashed, "\delta^i"] & \mathrm{H}^{i+1}(X^\bullet) \arrow[r, dashed] & \mathrm{H}^{i+1}(Y^\bullet) \arrow[r, dashed] & \mathrm{H}^{i+1}(Z^\bullet)
\end{tikzcd}$$

which proves (i) and (ii).

We now prove (iii). Suppose we have a distinguished triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X[1]^\bullet$$

in $\mathbf{K}(\mathcal{A})$. Shifting this to the left, we still have a distinguished triangle which hence has an isomorphism with a mapping cone triangle of some morphism $u\colon U^\bullet \to V^\bullet$, and shifting to the right again provides us with the isomorphism

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] \arrow[d, "\wr"] & Y^\bullet \arrow[r, "g"] \arrow[d, "\wr"] & Z^\bullet \arrow[r, "h"] \arrow[d, "\wr"] & X[1]^\bullet \arrow[d, "\wr"] \\
V^\bullet \arrow[r, "\alpha_u"] & C_u^\bullet \arrow[r, "\beta_u"] & U[1]^\bullet \arrow[r, "-u[1]"] & V[1]^\bullet
\end{tikzcd}$$

afterwhich we notice that

$$0 \longrightarrow V^\bullet \xrightarrow{\alpha_u} C_u^\bullet \xrightarrow{\beta_u} U[1]^\bullet \longrightarrow 0$$

is exact. Applying (i) to this together with the given isomorphism of triangles yields the exact sequence

$$\mathrm{H}^i(X^\bullet) \to \mathrm{H}^i(Y^\bullet) \to \mathrm{H}^i(Z^\bullet)$$

as desired. ■

Remark 4.16. See Theorem 5.33 for a similar kind of statement, albeit with a very different proof.

Corollary 4.17. *Let $\mathcal{A}$ be an Abelian category, and suppose we have an exact sequence*

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

in $\mathbf{C}(\mathcal{A})$. Then $\varphi^i := \begin{pmatrix} 0 & g^i \end{pmatrix}$ defines a quasi-isomorphism $\varphi\colon C_f^\bullet \to Z^\bullet$.

Proof. We have a diagram

$$\begin{tikzcd}
0 \arrow[r] & X^\bullet \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X^\bullet \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r, "g"] & Z^\bullet \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. Note, letting $z\colon 0 \to Z^\bullet$ be the zero map, that φ is the map obtained by first applying functoriality, giving $C_f^\bullet \to C_z^\bullet$, then realizing that $C_z^\bullet \cong 0 \oplus Z^\bullet \cong Z^\bullet$. In particular, we have a sequence of morphisms

$$0 \longrightarrow C^\bullet_{\mathrm{id}} \xrightarrow{\mathrm{id}\oplus f} C^\bullet_f \xrightarrow{0\oplus g} C^\bullet_z \longrightarrow 0.$$

In particular, it is then clear that this sequence is exact. We know that $C^\bullet_{\mathrm{id}}$ is homotopy equivalent to 0, and therefore that $\mathrm{H}^i(C^\bullet_{\mathrm{id}}) = 0$ for all $i \in \mathbb{Z}$. Then Proposition 4.15 gives us the exact sequence

$$0 \to \mathrm{H}^i(C^\bullet_f) \xrightarrow{\sim} \mathrm{H}^i(Z^\bullet) \to 0$$

where we used that $C^\bullet_z \cong Z^\bullet$. ■

4.3 THE DERIVED CATEGORY

Definition 4.18. Let $\mathcal{A}$ be an Abelian category. Define the full subcategory of $\mathbf{K}(\mathcal{A})$ given by

$$\mathcal{N} = \mathcal{N}_{\mathcal{A}} := \{X^\bullet \in \mathbf{K}(\mathcal{A}) \mid X \text{ is quasi-isomorphic to } 0\}.$$

Also define the system in $\mathbf{C}(\mathcal{A})$ of morphisms

$$\mathrm{Qis} = \mathrm{Qis}_{\mathcal{A}} := \{\text{quasi-isomorphisms in } \mathbf{C}(\mathcal{A})\}.$$

Proposition 4.19. *$\mathcal{N}$ is a null system in $\mathbf{K}(\mathcal{A})$ and $\mathcal{S}(\mathcal{N}) = \mathrm{Qis}$ in $\mathbf{K}(\mathcal{A})$.*

Proof. It is clear that $\mathcal{N}$ is closed under isomorphism and that $0 \in \mathcal{N}$. That $\mathcal{N}$ is closed under shifting is also clear, since $\mathrm{H}^i(X[1]^\bullet) = \mathrm{H}^{i+1}(X^\bullet)$. Thus, we have (N0)–(N2). To see that (N3) holds, consider a distinguished triangle

$$X^\bullet \to Y^\bullet \to Z^\bullet \to X[1]^\bullet$$

with $X^\bullet$ and $Z^\bullet$ in $\mathcal{N}$. Then the exact sequence

$$0 \longrightarrow \mathrm{H}^i(Y^\bullet) \longrightarrow 0$$

in $\mathcal{A}$ implies that $\mathrm{H}^i(Y^\bullet) = 0$ for all i. Thus $Y^\bullet$ is also in $\mathcal{N}$. This proves that $\mathcal{N}$ is a null system.

That $\mathrm{Qis} = \mathcal{S}(\mathcal{N})$ follows from the fact that the functors H^i are cohomological. In particular, given a distinguished triangle $X^\bullet \to Y^\bullet \to Z^\bullet \to X[1]^\bullet$ where $Z^\bullet \in \mathcal{N}$, we get an exact sequence

$$0 \to \mathrm{H}^i(X^\bullet) \to \mathrm{H}^i(Y^\bullet) \to 0$$

after shifting to the left. Thus the map $X \to Y$ is a quasi-isomorphism. Conversely, if we have a quasi-isomorphism $X^\bullet \to Y^\bullet$, then applying the five lemma (Corollary 2.35) we obtain isomorphisms

$$\begin{tikzcd}
\mathrm{H}^i(X^\bullet) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{H}^i(Y^\bullet) \arrow[r] \arrow[d, equal] & \mathrm{H}^i(Z^\bullet) \arrow[r] \arrow[d, dashed, "\wr"] & \mathrm{H}^{i+1}(X^\bullet) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{H}^{i+1}(Y^\bullet) \arrow[d, equal] \\
\mathrm{H}^i(X^\bullet) \arrow[r, "\sim"] & \mathrm{H}^i(Y^\bullet) \arrow[r] & 0 \arrow[r] & \mathrm{H}^{i+1}(X^\bullet) \arrow[r, "\sim"] & \mathrm{H}^{i+1}(Y^\bullet)
\end{tikzcd}$$

thereby showing that $\mathrm{H}^i(Z^\bullet) = 0$. Therefore, $\mathrm{Qis}_{\mathcal{A}} = \mathcal{S}(\mathcal{N}_{\mathcal{A}})$. ■

Remark 4.20. It follows that while Qis may not be a multiplicative system in $\mathbf{C}(\mathcal{A})$, it is a multiplicative system in $\mathbf{K}(\mathcal{A})$.

Finally, we can write down the actual definition of the derived category.

Definition 4.21. Let $\mathcal{A}$ be an Abelian category. The *derived category* $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ is the Verdier quotient $\mathbf{K}(\mathcal{A})/\mathcal{N}_{\mathcal{A}}$.

Remark 4.22. By the above proposition, we see that $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})_{\mathrm{Qis}}$.

We should now convince ourselves that this is the correct category. To do this, we will show that it is exactly the localization $\mathbf{C}(\mathcal{A})_{\mathrm{Qis}}$. In order to do that, we will first take a diversion into identifying a universal property of $\mathbf{K}(\mathcal{A})$. First, we define the *cylinder* of a chain complex.

Definition 4.23. Let $A^\bullet \in \mathbf{C}(\mathcal{A})$ be a chain complex. The *cylinder* $\mathbf{I}(A)^\bullet$ is the chain complex given by

$$\mathbf{I}(A)^i := A^i \oplus A^{i+1} \oplus A^i, \quad d^i = \begin{pmatrix} d_A^i & \mathrm{id}_{A^{i+1}} & 0 \\ 0 & -d_A^{i+1} & 0 \\ 0 & -\mathrm{id}_{A^{i+1}} & d_A^i \end{pmatrix}.$$

Remark 4.24. As usual, one should explicitly check that $d^{i+1} \circ d^i = 0$, but this is "a computation."

The reason this is called the cylinder is because it is analogous to cylinder objects in topology. In particular, in topology, a homotopy between two continuous maps $X \to Y$ is the same as a continuous map $X \times [0,1] \to Y$ which restricts to the original maps on $\{0\}$ and $\{1\}$. In other words, maps $X \times [0,1] \to Y$ are in canonical bijection with homotopy equivalent pairs of maps $X \to Y$ with a specified homotopy. The cylinder above also has this property.

Proposition 4.25. *Let $X^\bullet$ and $Y^\bullet$ be chain complexes in an Abelian category $\mathcal{A}$. Then we have a bijection*

$$\mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(\mathbf{I}(X^\bullet), Y^\bullet) \xrightarrow{\sim} \{(f, g, \eta) \mid f, g \in \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X^\bullet, Y^\bullet),\ f \sim_{\mathrm{h}} g \text{ via } \eta\colon f \Rightarrow g\}$$

given by $(h : \mathbf{I}(X)^\bullet \to Y^\bullet) \mapsto (h_1, h_3, h_2)$, where h_j, $j \in \{1, 2, 3\}$, refers to the maps in the decomposition

$$h^i = \begin{pmatrix} h_1^i & h_2^i & h_3^i \end{pmatrix}.$$

In particular, h_1 and h_3 define maps of chain complexes $X^\bullet \to Y^\bullet$.

Proof. Write $f^i = h_1^i$, $\eta^{i+1} = h_2^i$, and $g^i = h_3^i$. The condition that h is a morphism of chain complexes means that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathbf{I}(X)^i \arrow[r, "d^i"] \arrow[d, "h^i"'] & \mathbf{I}(X)^{i+1} \arrow[r] \arrow[d, "h^{i+1}"] & \cdots \\ \cdots \arrow[r] & Y^i \arrow[r, "d^i"] & Y^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

commutes. Writing out the two possible compositions explicitly, we have

$$d^i \circ h^i = d^i \circ \begin{pmatrix} f^i & \eta^{i+1} & g^i \end{pmatrix} = \begin{pmatrix} d^i \circ f^i & d^i \circ \eta^{i+1} & d^i \circ g^i \end{pmatrix}$$

and

$$\begin{aligned} h^{i+1} \circ d^i &= \begin{pmatrix} f^{i+1} & \eta^{i+2} & g^{i+1} \end{pmatrix} \begin{pmatrix} d_X^i & \mathrm{id}_{X^{i+1}} & 0 \\ 0 & -d_X^{i+1} & 0 \\ 0 & -\mathrm{id}_{X^{i+1}} & d_A^i \end{pmatrix} \\ &= \begin{pmatrix} f^{i+1} \circ d^i & f^{i+1} - \eta^{i+2} \circ d^{i+1} - g^{i+1} & g^{i+1} \circ d^i \end{pmatrix}. \end{aligned}$$

Since these two are equal, we have that $f^{i+1} \circ d^i = d^i \circ f^i$, $g^{i+1} \circ d^i = d^i \circ g^i$, and that

$$d^i \circ \eta^{i+1} = f^{i+1} - \eta^{i+2} \circ d^{i+1} - g^{i+1} \implies f^{i+1} - g^{i+1} = d^i \circ \eta^{i+1} + \eta^{i+2} \circ d^{i+1}.$$

That is, we obtain two maps of chain complexes $f, g\colon X^\bullet \to Y^\bullet$ and a homotopy $\eta\colon f \Rightarrow g$. ■

Lemma 4.26. *Let $X^\bullet$ be a chain complex in an Abelian category $\mathcal{A}$. Then the canonical inclusions $i_1, i_3 \colon X^\bullet \to \mathbf{I}(X)^\bullet$ of $X^\bullet$ to the first and third components are homotopy equivalences with inverse $\begin{pmatrix}\mathrm{id} & 0 & \mathrm{id}\end{pmatrix}$.*

Proof. We prove this for i_1, since the proof for i_3 is similar. Let $\pi = \begin{pmatrix}\mathrm{id} & 0 & \mathrm{id}\end{pmatrix}$. Then it is easily seen that $\pi \circ i_1 = \mathrm{id}_{X^\bullet}$. To see that $i_1 \circ \pi \sim_{\mathrm{h}} \mathrm{id}_{\mathbf{I}(X)^\bullet}$, consider the map

$$\eta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} d^{i-1} \circ \eta^i + \eta^{i+1} \circ d^i &= \begin{pmatrix} d_X^{i-1} & \mathrm{id}_{X^i} & 0 \\ 0 & -d_X^i & 0 \\ 0 & -\mathrm{id}_{X^i} & d_X^{i-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_X^{i} & \mathrm{id}_{X^{i+1}} & 0 \\ 0 & -d_X^{i+1} & 0 \\ 0 & -\mathrm{id}_{X^{i+1}} & d_X^{i} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -\mathrm{id} \\ 0 & 0 & d_X^i \\ 0 & 0 & \mathrm{id} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{id} & -d_X^i \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\mathrm{id} \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} = \mathrm{id} - i_1 \circ \pi \end{aligned}$$

so η defines a homotopy $\mathrm{id} \Rightarrow i_1 \circ \pi$. ■

Remark 4.27. This lemma has an analogue in topology, which may be considered the motivation. Let X be a topological space, and again consider the cylinder $X \times [0,1]$. This has a projection $X \times [0,1] \to X$ by simply forgetting the second variable, and there are two maps $X \to X \times [0,1]$ given by $x \mapsto (x,0)$ and $x \mapsto (x,1)$. It is then true that these maps provide two different homotopy equivalences between X and $X \times [0,1]$.

Theorem 4.28. *Let $\mathcal{A}$ be an Abelian category, and let $U \colon \mathbf{C}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ be the obvious functor. Then for any category $\mathcal{E}$ with a functor $F \colon \mathbf{C}(\mathcal{A}) \to \mathcal{E}$ such that F sends homotopy equivalences to isomorphisms, there exists a unique functor $\hat{F} \colon \mathbf{K}(\mathcal{A}) \to \mathcal{E}$ such that $F = \hat{F} \circ U$.*

Proof. We already know that this is true if we have $F(f) = F(g)$ whenever $f \sim_{\mathrm{h}} g$. Thus, it suffices to show that any F sending homotopy equivalences to isomorphisms satisfies this. Let $f, g \colon X^\bullet \to Y^\bullet$ be two homotopy equivalent maps. By Proposition 4.25, this produces a unique map $h : \mathbf{I}(X)^\bullet \to Y^\bullet$ (up to choice of homotopy). We then note that there is a commutative diagram

$$\begin{tikzcd}
X^\bullet \arrow[d, shift left, "i_1"] \arrow[dr, "f"] & \\
\mathbf{I}(X)^\bullet \arrow[u, shift left, "\pi"] \arrow[r, "h"] \arrow[d, shift right, "\pi"'] & Y^\bullet \\
X^\bullet \arrow[u, shift right, "i_3"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

where $\pi = \begin{pmatrix}\mathrm{id} & 0 & \mathrm{id}\end{pmatrix}$ and i_k is the inclusion to the kth component. Lemma 4.26 then tells us that π is a homotopy equivalence, so $F(\pi)$ is an isomorphism. Furthermore, $\pi \circ i_1 = \pi \circ i_3 = \mathrm{id}$, so we know that both $F(i_1)$ and $F(i_3)$ are inverses to $F(\pi)$. Therefore, $F(i_1) = F(i_3)$. Applying F to the above commutative diagram then yields

$$F(f) = F(h) \circ F(i_1) = F(h) \circ F(i_3) = F(g).$$

This completes the proof. ■

We name this a theorem because it now yields us our desired equivalence very easily.

Corollary 4.29. *Let $\mathcal{A}$ be an Abelian category. Then there is a canonical isomorphism of categories* $\mathbf{C}(\mathcal{A})_{\mathrm{Qis}} \cong \mathbf{D}(\mathcal{A})$.

Proof. We show that $\mathbf{D}(\mathcal{A})$ satisfies the universal property of $\mathbf{C}(\mathcal{A})_{\mathrm{Qis}}$. Consider a functor $F\colon \mathbf{C}(\mathcal{A}) \to \mathcal{E}$ sending quasi-isomorphisms to isomorphisms. Note that homotopy equivalences are necessarily quasi-isomorphisms by the functoriality of cohomology, and therefore F factors through a unique functor $\hat{F} : \mathbf{K}(\mathcal{C}) \to \mathcal{E}$. The functor $\hat{F}$ now sends quasi-isomorphisms to isomorphisms, and therefore—since we know that $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})_{\mathrm{Qis}}$—by the universal property of $\mathbf{D}(\mathcal{A})$, this factors uniquely through $\hat{F}_{\mathrm{Qis}}\colon \mathbf{D}(\mathcal{A}) \to \mathcal{E}$. This provides the unique factorization of F through $\mathbf{D}(\mathcal{A})$. ■

With this, we can be quite convinced that $\mathbf{D}(\mathcal{A})$, defined as the Verdier quotient $\mathbf{K}(\mathcal{A})/\mathcal{N}_{\mathcal{A}}$, is the correct object to be working with. Having satisfied that, we can move on to actually studying the derived category. Here, our primary aim is to identify $\mathcal{A}$ as living inside $\mathbf{D}(\mathcal{A})$ as a full additive subcategory. To make life easier here, we will to backtrack a little and inspect some natural functors defined on $\mathbf{C}(\mathcal{A})$, namely the *truncation* functors. These will appear in an abstract form later in our discussion of t-structures (see Section 5).

Definition 4.30. Let $\mathcal{A}$ be an Abelian category, and let $X^\bullet \in \mathbf{C}(\mathcal{A})$. For any $n \in \mathbb{Z}$, we define the chain complexes

$$\begin{array}{lcccccccccccc} \tau^{\leq n} X^\bullet := & \cdots & \to & X^{n-2} & \to & X^{n-1} & \to & \ker d_X^n & \to & 0 & \to & 0 & \to \cdots \\ \tau^{\geq n} X^\bullet := & \cdots & \to & 0 & \to & 0 & \to & \operatorname{coker} d_X^{n-1} & \to & X^{n+1} & \to & X^{n+2} & \to \cdots \end{array}$$

This data defines two additive functors $\tau^{\leq n}, \tau^{\geq n}\colon \mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{A})$.

Remark 4.31. In particular, for a morphism $f : X^\bullet \to Y^\bullet$, we have that the truncations of f have the same components as f itself when that statement makes sense, and is zero otherwise. For $\tau^{\leq n}$, for example, we have $(\tau^{\leq n} f)^i = f^i$ for $i < n$, $(\tau^{\leq n} f)^i = 0$ for $i > n$, and that $(\tau^{\leq n} f)^n$ is the canonical induced map $\ker d_X^n \to \ker d_Y^n$. That this is an additive functor is clear.

We will now collect some propositions about these truncation functors.

Proposition 4.32. *Consider a morphism of chain complexes $f : X^\bullet \to Y^\bullet$. Then we have natural isomorphisms* $\mathrm{H}^i(\tau^{\leq n} X^\bullet) = \mathrm{H}^i(X^\bullet)$ *for* $i \leq n$, $\mathrm{H}^i(\tau^{\leq n} X^\bullet) = 0$ *for* $i > n$, $\mathrm{H}^i(\tau^{\geq n} X^\bullet) = \mathrm{H}^i(X^\bullet)$ *for* $i \geq n$, *and* $\mathrm{H}^i(\tau^{\geq n} X^\bullet) = 0$ *for* $i < n$. *Furthermore, the morphisms which $\tau^{\leq n} f$ and $\tau^{\geq n} f$ induce on cohomology satisfy*

$$\mathrm{H}^i(\tau^{\leq n} f) = \begin{cases} \mathrm{H}^i(f) & \text{for } i \leq n, \\ 0 & \text{for } i > n. \end{cases}$$

$$\mathrm{H}^i(\tau^{\geq n} f) = \begin{cases} \mathrm{H}^i(f) & \text{for } i \geq n, \\ 0 & \text{for } i < n. \end{cases}$$

Proof. We provide a proof for $\tau^{\leq n}$, since the other one is dual. The first statement follows by the observation that for $i < n$, $(\tau^{\leq n} X^\bullet) i = X^i$ and, in particular, the differentials are unaffected. Therefore, $\mathrm{H}^i(\tau^{\leq n} X^\bullet) = \mathrm{H}^i(X^\bullet)$ in those cases. For $i > n$, the truncated complex is simply zero, so there $\mathrm{H}^i(\tau^{\leq n} X^\bullet) = 0$. For $i = n$, we observe that since the differential $d^n_{\tau^{\leq n} X}$ is exactly d_X^n (although factored through the inclusion $\ker d_X^n \hookrightarrow X^n$), we have the diagram

$$\begin{tikzcd}
 & & & \mathrm{H}^n(X^\bullet) \arrow[dd, leftrightarrow, "\wr"] \\
X^{n-1} \arrow[r, two heads] & \operatorname{im} d^{n-1} \arrow[r, hook] & \ker d^n \arrow[ur, two heads] \arrow[dr, two heads] & \\
 & & & \mathrm{H}^n(\tau^{\leq n} X^\bullet)
\end{tikzcd}$$

We now prove the second statement. Note that it is trivial for $i < n$. In particular, the map $\mathrm{H}^i(f)$ depends only on the images of d_X^{i-1} and d_Y^{i-1}, the kernels of d_X^i and d_Y^i, and map f^i. When $i < n$, these parameters are unchanged after truncation since then $(\tau^{\leq n} f)^i = f^i$, and therefore $\mathrm{H}^i(\tau^{\leq n} f) = \mathrm{H}^i(f)$. On the other hand, the statement is trivial for $i > n$ because then every map involved is zero. Thus the only seemingly non-trivial case is $n = i$. However, here we observe that when we construct the map on cohomology for $\tau^{\leq n} f$, we are actually using the diagram

$$\begin{tikzcd}
X^{n-1} \arrow[r, two heads] \arrow[d, "f^{n-1}"] & \operatorname{im} d_X^{n-1} \arrow[r, hook] \arrow[d] & \ker d_X^n \arrow[r, two heads] \arrow[d, "(\tau^{\leq n} f)^n"] & \mathrm{H}^n(X^\bullet) \arrow[d] \\
Y^{n-1} \arrow[r, two heads] & \operatorname{im} d_Y^{n-1} \arrow[r, hook] & \ker d_B^n \arrow[r, two heads] & \mathrm{H}^n(Y^\bullet)
\end{tikzcd}$$

where $(\tau^n f)^n$ is exactly the same map induced on the kernels as in the comparable diagram for f^n. Since the diagram is the same, we see that $\mathrm{H}^n(\tau^{\leq n} f) = \mathrm{H}^n(f)$. ■

Proposition 4.33. *Suppose we have two maps of chain complexes $f, g\colon X^\bullet \to Y^\bullet$ where $f \sim_{\mathrm{h}} g$. Then $\tau^{\leq n} f \sim_{\mathrm{h}} \tau^{\leq n} g$ and $\tau^{\geq n} f \sim_{\mathrm{h}} \tau^{\geq n} g$. Thus the truncation functors induce unique functors $\tau^{\leq n}, \tau^{\geq n}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$.*

Proof. Since the truncation functors are additive, it suffices to show the result when $f \sim_{\mathrm{h}} 0$. Since the proofs are dual, we show the result only for $\tau^{\leq n}$.

Suppose $f \sim_{\mathrm{h}} 0$. Then, by definition, we have a homotopy $\eta\colon f \Rightarrow 0$. Note then that taking the components η^i for $i < n$, 0 for $i > n$, and the induced map $\ker d_X^n \hookrightarrow X^n \to Y^{n-1}$ for $i = n$ gives a homotopy $\sigma\colon \tau^{\leq n} f \Rightarrow 0$. In particular, from the fact that η is a homotopy, we have that $f^n = \eta^{n+1} \circ d_X^n + d_Y^{n-1} \circ \eta^n$. Restricting to the kernel means that the left term is just the zero map, and so we have an equality

$$(\tau^{\leq n} f)^n = 0 + d^{n-1} \circ \sigma^n = \sigma^{n+1} \circ d^n + d^{n-1} \circ \sigma^n.$$

This proves the proposition. ■

Proposition 4.34. *For all $m, n \in \mathbb{Z}$, there is a natural isomorphism $\tau^{\leq m} \circ \tau^{\geq n} \cong \tau^{\geq n} \circ \tau^{\leq m}$. Furthermore, the cohomology functor H^n is the same as the functor which sends $X^\bullet$ to the nth term in the complex $\tau^{\leq n} \tau^{\geq n} X^\bullet$.*

Proof. When $m < n$, both the double truncations simply produce the zero complex. When $m > n$, the two operations do not touch any overlapping parts of the complex, so the result is clear. Thus, what remains is $m = n$. Both $\tau^{\leq n} \tau^{\geq n} X^\bullet$ and $\tau^{\geq n} \tau^{\leq n} X^\bullet$ are complexes concentrated in degree n. Furthermore, the nth terms are exactly $\ker(\operatorname{coker} d_X^{n-1} \to X^{n+1})$ and $\operatorname{coker}(X^{n-1} \to \ker d^n)$, which we know from Section 2 are both exactly the cohomology of $X^\bullet$ at n. The last statement follows trivially from this also. ■

Remark 4.35. We did not specify the categories we were working inside in the above proposition. This is because it holds both for $\mathbf{C}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$, and the proof is the same in both cases.

Proposition 4.36. *For all $n \in \mathbb{Z}$, the functors $\mathrm{H}^n : \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ and $\tau^{\leq n}, \tau^{\geq n} : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ induce functors $\mathrm{H}^n : \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ and $\tau^{\leq n}, \tau^{\geq n} : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$. Furthermore, H^n is cohomological, and a morphism $f\colon X^\bullet \to Y^\bullet$ in $\mathbf{D}(\mathcal{A})$ is an isomorphism if and only if $\mathrm{H}^i(f)$ is an isomorphism for all $i \in \mathbb{Z}$.*

Proof. Since $\mathrm{H}^i(X^\bullet) = 0$ for all $X^\bullet \in \mathcal{N}_\mathcal{A}$, it immediately induces the given functor $\mathbf{D}(\mathcal{A}) \to \mathcal{A}$. That this is cohomological is now automatic. Similarly, if a morphism $f\colon X^\bullet \to Y^\bullet$ in $\mathbf{K}(\mathcal{A})$ is a quasi-isomorphism, then so are $\tau^{\leq n} f$ and $\tau^{\geq n} f$. Therefore, these induce functors $\mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.

To prove the final statement, let $f\colon X^\bullet \to Y^\bullet$ be a morphism in $\mathbf{D}(\mathcal{A})$. Rewrite this as a roof gs^{-1}, with $s\colon X'^\bullet \to X^\bullet$ a quasi-isomorphism and $g\colon X'^\bullet \to Y^\bullet$ a morphism in $\mathbf{K}(\mathcal{A})$. Then, since $f = Q(g) \circ Q(s)^{-1}$ and hence $\mathrm{H}^i(f) = \mathrm{H}^i(g) \circ \mathrm{H}^i(s)^{-1}$, we have

$$\forall i \in \mathbb{Z}, \mathrm{H}^i(f) \text{ is an iso.} \iff \forall i \in \mathbb{Z}, \mathrm{H}^i(g) \text{ is an iso.} \iff g \text{ is a qis.} \iff f \text{ is an iso.}$$

which completes the proof. ■

So far, we have dealt with modifications of the category $\mathbf{C}(\mathcal{A})$. We have neglected mentioning that there are various full subcategories of this which are of interest.

Definition 4.37. Let $\mathcal{A}$ be an Abelian category, and let $n \in \mathbb{Z}$. Define the full subcategories

$$\mathbf{C}^{\leq n}(\mathcal{A}) := \{X^\bullet \in \mathbf{C}(\mathcal{A}) \mid X^i = 0 \text{ for } i > n\}, \quad \mathbf{C}^{\geq n}(\mathcal{A}) := \{X^\bullet \in \mathbf{C}(\mathcal{A}) \mid X^i = 0 \text{ for } i < n\}.$$

Furthermore, define from these the additional full subcategories

$$\mathbf{C}^-(\mathcal{A}) = \bigcup_{n \in \mathbb{Z}} \mathbf{C}^{\leq n}(\mathcal{A}), \quad \mathbf{C}^+(\mathcal{A}) = \bigcup_{n \in \mathbb{Z}} \mathbf{C}^{\geq n}(\mathcal{A}), \quad \mathbf{C}^b(\mathcal{A}) = \mathbf{C}^-(\mathcal{A}) \cap \mathbf{C}^+(\mathcal{A}).$$

Letting $*$ be any of the above decorations, define $\mathbf{K}^*(\mathcal{A})$ in the obvious way. For $* \in \{-, +, b\}$, define $\mathbf{D}^*(\mathcal{A}) = \mathbf{K}^*(\mathcal{A})/(\mathcal{N}_\mathcal{A} \cap \mathbf{K}^*(\mathcal{A}))$.

Remark 4.38. Note that the last definition is actually fine. In particular, the whole theory we have built for $\mathbf{K}(\mathcal{A})$ also works for $\mathbf{K}^*(\mathcal{A})$, at least when $* \in \{-, +, b\}$. Notably, the latter is a triangulated category. When we appropriately restrict the truncation functors to these, we end up with functors $\tau^{\leq n}\colon \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})$ and $\tau^{\geq n}\colon \mathbf{D}^-(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})$.

Remark 4.39. Observe also that if $X^\bullet \in \mathbf{C}^{\leq n}(\mathcal{A})$, then the obvious map $\tau^{\leq n} X^\bullet \to X^\bullet$ is a quasi-isomorphism. Dually, for $Y^\bullet \in \mathbf{C}^{\geq n}(\mathcal{A})$, the obvious map $Y^\bullet \to \tau^{\geq n} Y^\bullet$ is a quasi-isomorphism.

We now come to a key proposition which we need for the future. It gives us a simple description of morphisms in the derived category in a specific restricted context.

Proposition 4.40. [KS06, Prop. 13.1.8] *Let $n \in \mathbb{Z}$, let $X^\bullet \in \mathbf{K}^{\leq n}(\mathcal{A})$, and let $Y^\bullet \in \mathbf{K}^{\geq n}(\mathcal{A})$. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_\mathcal{A}(\mathrm{H}^n(X^\bullet), \mathrm{H}^n(Y^\bullet)).$$

Proof sketch. Lift $X^\bullet$ and $Y^\bullet$ to $\mathbf{C}(\mathcal{A})$, and consider two morphisms $f, g : X^\bullet \to Y^\bullet$ (i.e. not up to homotopy). Note that any homotopy $\eta\colon f \Rightarrow g$ must consist of only the zero maps, and therefore $f = g$. In other words, we have a natural isomorphism

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X^\bullet, Y^\bullet).$$

We then compute

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X^\bullet, Y^\bullet) &\cong \{f \in \mathrm{Hom}_\mathcal{A}(X^n, Y^n) \mid f \circ d_X^{n-1} = 0,\ d_Y^n \circ f = 0\} \\ &\cong \mathrm{Hom}_\mathcal{A}(\operatorname{coker} d_X^{n-1}, \ker d_Y^n) \\ &\cong \mathrm{Hom}_\mathcal{A}(\mathrm{H}^n(X^\bullet), \mathrm{H}^n(Y^\bullet)). \end{aligned}$$

We now just have to know that in this particular case, $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X^\bullet, Y^\bullet)$. To prove this, we will use [KS06, Prop. 2.5.2]. In other words, we need $\mathrm{Qis}^{Y^\bullet/} \cap K^{\geq n}(\mathcal{A})^{Y^\bullet/}$ to be cofinal to $\mathrm{Qis}^{Y^\bullet/}$ when $Y^\bullet \in \mathbf{K}^{\geq n}(\mathcal{A})$ (which is the situation we are in). This is easily seen using truncation and some general propositions in abstract nonsense allowing us to deduce cofinality from the fact that, for any quasi-isomorphism $Y^\bullet \to Z^\bullet$, we very obviously have a morphism

$$(Y^\bullet \to Z^\bullet) \to (Y^\bullet \to \tau^{\geq n} Z^\bullet).$$

The full argument can be found by chasing references in [KS06, Lemma 13.1.7]. From the cofinality, we can compute

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X^\bullet, Y^\bullet) &= \varinjlim_{(Y^\bullet \to Y'^\bullet) \in \mathrm{Qis}^{Y^\bullet/}} \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X^\bullet, Y'^\bullet) \\
&\cong \varinjlim_{(Y^\bullet \to Y'^\bullet) \in \mathrm{Qis}^{Y^\bullet/} \cap K^{\geq n}(\mathcal{A})^{Y^\bullet/}} \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X^\bullet, Y'^\bullet) \\
&\cong \varinjlim_{Y'} \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^n(X^\bullet), \mathrm{H}^n(Y'^\bullet)) \\
&\cong \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^n(X^\bullet), \mathrm{H}^n(Y^\bullet)),
\end{aligned}$$

where the first $\cong$ is where we use cofinality. This completes the argument. □

Theorem 4.41. *Let $\mathcal{A}$ be an Abelian category. Then the composition $\mathcal{A} \to \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ induces an equivalence between $\mathcal{A}$ and the full subcategory of $\mathbf{D}(\mathcal{A})$ spanned by those complexes $X^\bullet$ for which $\mathrm{H}^n(X^\bullet) = 0$ for all $n \neq 0$.*

Proof. We show that the functor $F : \mathcal{A} \to \mathbf{D}(\mathcal{A})$ sending X to the complex concentrated in degree 0 is fully faithful and essentially surjective. That it is fully faithful is a corollary of the preceding Proposition 4.40. In particular, we have that

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(F(X), F(Y)) \cong \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(F(X)), \mathrm{H}^0(F(Y))) \cong \mathrm{Hom}_{\mathcal{A}}(X, Y).$$

To see that F is essentially surjective, consider a complex $X^\bullet$ such that $\mathrm{H}^i(X^\bullet) = 0$ for all $i \neq 0$. Then we note that $X^\bullet \cong \tau^{\leq 0} X^\bullet \cong \tau^{\geq 0} \tau^{\leq 0} X^\bullet \cong F(\mathrm{H}^0(X^\bullet))$. ■

Remark 4.42. A more complete statement would also identify the categories $\mathbf{D}^*(\mathcal{A})$, $* \in \{-, +, b\}$, with certain (obvious) full subcategories of $\mathbf{D}(\mathcal{A})$. We will neglect actually proving this. The full argument can be found in [KS06, Prop. 13.1.12], and it is really a result about comparing localizations of (full) subcategories with localizations of an ambient category.

Finally, we will want a nice way to produce distinguished triangles in the derived category.

Theorem 4.43. *Let $\mathcal{A}$ be an Abelian category. Suppose we have an exact sequence*

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$$

in $\mathbf{C}(\mathcal{A})$. Then there is a distinguished triangle

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X[1]^\bullet$$

in $\mathbf{D}(\mathcal{A})$.

Proof. By Corollary 4.17, we have a quasi-isomorphism $C_f^\bullet \to Z^\bullet$. This allows us to define a map $Z^\bullet \to X[1]^\bullet$ in $\mathbf{D}(\mathcal{A})$ and hence define a triangle

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X[1]^\bullet$$

which must be distinguished since it is isomorphic to a distinguished triangle. ■

5 t-Structures

This section is dedicated to studying t-structures along with some basic properties and constructions related to them. We mainly follow the exposition in [KS94], but also include content on extensions and gluing which are not present there. It should also be noted that the proofs in the aforementioned book are largely based on the ones in [BBDG18]. We begin with a motivating example for the definition of a t-structure.

Let $\mathcal{A}$ be an Abelian category. We know that the derived category $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ has the structure of a triangulated category. However, the triangulated structure on it does not carry all the information arising from the chain complexes themselves. Indeed, there are a number of nice features which do not follow just from the triangles in $\mathbf{D}(\mathcal{A})$. To see this, note that we have two full subcategories

$$\mathbf{D}(\mathcal{A})^{\leq 0} = \{A^\bullet \in \mathbf{D}(\mathcal{A}) \mid \forall i > 0,\ \mathrm{H}^i(A^\bullet) = 0\}, \quad \mathbf{D}(\mathcal{A})^{\geq 0} = \{A^\bullet \in \mathbf{D}(\mathcal{A}) \mid \forall i < 0,\ \mathrm{H}^i(A^\bullet) = 0\}.$$

Letting $\mathbf{D}(\mathcal{A})^{\leq n} := \mathbf{D}(\mathcal{A})^{\leq 0}[-n]$, $\mathbf{D}(\mathcal{A})^{\geq n} := \mathbf{D}(\mathcal{A})^{\geq 0}[-n]$, we see that

$$\mathbf{D}(\mathcal{A})^{\leq n} = \{A^\bullet \in \mathbf{D}(\mathcal{A}) \mid \forall i > n,\ \mathrm{H}^i(A^\bullet) = 0\}, \quad \mathbf{D}(\mathcal{A})^{\geq n} = \{A^\bullet \in \mathbf{D}(\mathcal{A}) \mid \forall i < n,\ \mathrm{H}^i(A^\bullet) = 0\}.$$

These subcategories verify a few relations. For example, let $X^\bullet \in \mathbf{D}(\mathcal{A})^{\leq 0}$ and $Y^\bullet \in \mathbf{D}(\mathcal{A})^{\geq 1}$. Then we note that, by Proposition 4.40 after replacing $X^\bullet$ by $\tau^{\leq 0}X^\bullet$ and $Y^\bullet$ by $\tau^{\geq 1}Y^\bullet$ (see Remark 4.39; the point is that now we have $X^i = 0$ for $i > 0$, $Y^i = 0$ for $i < 1$), we have

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(X^\bullet), \mathrm{H}^0(Y^\bullet)) \cong \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(X^\bullet), 0) = 0.$$

That is, there are no maps from $\mathbf{D}(\mathcal{A})^{\leq 0}$ to $\mathbf{D}(\mathcal{A})^{\geq 1}$.

Furthermore, for an arbitrary complex $X^\bullet \in \mathbf{D}(\mathcal{A})$, we have an exact sequence

$$0 \longrightarrow \tau^{\leq 0}X^\bullet \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow 0$$

where $Y^\bullet$ is the cokernel of $\tau^{\leq 0}X^\bullet \to X^\bullet$. By Theorem 4.43, this establishes a distinguished triangle

$$\tau^{\leq 0}X^\bullet \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow (\tau^{\leq 0}X^\bullet)[1]$$

in $\mathbf{D}(\mathcal{A})$. Now observe that $Y^\bullet$ is quasi-isomorphic to $\tau^{\geq 1}X^\bullet$. Indeed, we have a morphism of chain complexes $\tau^{\geq 1}X^\bullet \to Y^\bullet$ given by

$$\begin{tikzcd}
\tau^{\geq 1}X^\bullet : \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & \operatorname{coker} d_X^0 \arrow[r] \arrow[d, two heads] & X^1 \arrow[r] \arrow[d, equal] & \cdots \\
Y^\bullet : & \cdots \arrow[r] & 0 \arrow[r] & \operatorname{im} d_X^0 \arrow[r] & X^1 \arrow[r] & \cdots
\end{tikzcd}$$

where the epimorphism is described in Section 2.2. This morphism is a quasi-isomorphism, so we see that $Y^\bullet \cong \tau^{\geq 1}X^\bullet$ in $\mathbf{D}(\mathcal{C})$, and in particular that that we have an isomorphism of triangles

$$\begin{tikzcd}
\tau^{\leq 0}X^\bullet \arrow[r] \arrow[d, equal] & X^\bullet \arrow[r] \arrow[d, equal] & \tau^{\geq 1}X^\bullet \arrow[r] \arrow[d, "\wr"] & (\tau^{\leq 0}X^\bullet)[1] \arrow[d, equal] \\
\tau^{\leq 0}X^\bullet \arrow[r] & X^\bullet \arrow[r] & Y^\bullet \arrow[r] & (\tau^{\leq 0}X^\bullet)[1]
\end{tikzcd}$$

so that the triangle at the top is distinguished.

5.1 t-STRUCTURES & TRUNCATION FUNCTORS

The above computation serves to motivate the following definition, which aims to generalize the situation to triangulated categories:

Definition 5.1. Let $\mathcal{D}$ be a triangulated category. A *t-structure* on $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories of $\mathcal{D}$, called the *aisle* and *co-aisle*, such that

(T1) if $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 0}$ then $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$,

(T2) $\mathcal{D}^{\leq 0}[1] \subseteq \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}[-1] \subseteq \mathcal{D}^{\geq 0}$, and

(T3) if $X \in \mathcal{D}$, then there is a distinguished triangle $X' \to X \to X'' \to X'[1]$ where $X' \in \mathcal{D}^{\leq 0}$ and $X'' \in \mathcal{D}^{\geq 0}[-1]$.

One writes $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$. The *heart* of $\mathcal{D}$ with respect to $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is

$$\mathcal{D}^{\heartsuit} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}.$$

Remark 5.2. Note that using the introduced notation, we see that $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$. Similarly, we see that whenever $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$, the only map $X \to Y$ is the zero map $X \to 0 \to Y$.

Remark 5.3. The requirement that the subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ are replete (i.e. strictly full) is, for many purposes, not entirely necessary. It is nonetheless extremely convenient.

Example 5.4. For any triangulated category $\mathcal{D}$, there are two immediate trivial t-structures one can put on it. In particular, one can set $\mathcal{D}^{\leq 0} = 0$ and $\mathcal{D}^{\geq 0} = \mathcal{D}$, or conversely one can set $\mathcal{D}^{\leq 0} = \mathcal{D}$ and $\mathcal{D}^{\geq 0} = 0$. Both of these are immediately verified to be t-structures.

Example 5.5. A more non-trivial and naturally occuring example of a t-structure is the one we have provided at the start of this section. In particular, the computation there essentially proves the following proposition:

Proposition 5.6. *Let $\mathcal{A}$ be an Abelian category. Then the pair $(\mathbf{D}(\mathcal{A})^{\leq 0}, \mathbf{D}(\mathcal{A})^{\geq 0})$ gives a t-structure on $\mathbf{D}(\mathcal{A})$. This t-structure is called the* standard t-structure *on $\mathbf{D}(\mathcal{A})$.*

Proof. That $\mathbf{D}(\mathcal{A})^{\leq 0}, \mathbf{D}(\mathcal{A})^{\geq 0}$ are stable under isomorphism is obvious. We have already checked conditions (T1) and (T3), so what remains is checking (T2). However, that $\mathbf{D}(\mathcal{A})^{\leq -1} \subseteq \mathbf{D}(\mathcal{A})^{\leq 0}$ and $\mathbf{D}(\mathcal{A})^{\geq 1} \subseteq \mathbf{D}(\mathcal{A})^{\geq 0}$ is completely obvious. ■

The constructions $X^\bullet \mapsto \tau^{\leq n} X^\bullet$, $X^\bullet \mapsto \tau^{\geq n} X^\bullet$ that we defined earlier are functorial: if we have complexes $X^\bullet, Y^\bullet \in \mathbf{D}(\mathcal{A})$ and a map $f^\bullet \colon X^\bullet \to Y^\bullet$, then it is easily seen that this induces a map $\tau^{\leq n} X^\bullet \to \tau^{\leq n} Y^\bullet$, given by $\tau^{\leq n} f^i = f^i$ for $i < n$, 0 for $i > n$, and the induced map $\ker d^n \to \ker d^n$ when $i = n$. The situation for $\tau^{\geq n}$ is similar. The point is, we see that we have functors

$$\tau^{\leq n} \colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})^{\leq n}, \quad \tau^{\geq n} \colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})^{\geq n}.$$

These functors are right (respectively, left) adjoints to the inclusions of $\mathbf{D}(\mathcal{A})^{\leq n}$, $\mathbf{D}(\mathcal{A})^{\geq n}$ into $\mathbf{D}(\mathcal{A})$. To see this, suppose we have $X^\bullet \in \mathbf{D}(\mathcal{A})$ and $Y^\bullet \in \mathbf{D}^{\leq n}(\mathcal{A})$. We have a distinguished triangle

$$\tau^{\leq n} X^\bullet \longrightarrow X^\bullet \longrightarrow \tau^{\geq n+1} X^\bullet \longrightarrow (\tau^{\leq n} X^\bullet)[1].$$

Since $\mathbf{D}(\mathcal{A})$ is a triangulated category, we know that $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(Y^\bullet, -)$ is a cohomological functor, so we have an exact sequence

$$\mathrm{Hom}(Y^\bullet, (\tau^{\geq n+1} X^\bullet)[-1]) \to \mathrm{Hom}(Y^\bullet, \tau^{\leq n} X^\bullet) \to \mathrm{Hom}(Y^\bullet, X^\bullet) \to \mathrm{Hom}(Y^\bullet, \tau^{\geq n+1} X^\bullet)$$

where we know that

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(Y^\bullet, (\tau^{\geq n+1}X^\bullet)[-1]) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(Y^\bullet, \tau^{\geq n+1}X^\bullet) = 0$$

so that we have the isomorphism

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(Y^\bullet, X^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(Y^\bullet, \tau^{\leq n}X^\bullet).$$

The result then follows from the Yoneda lemma. A similar computation can be done for $\tau^{\geq n}$.

These "truncation" functors appear more generally, and this is what we will prove now along with some properties of these. First, we need a lemma.

Lemma 5.7. *Let $\mathcal{C}$ be a category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory, and let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor. If $F(\mathcal{C}') \subseteq \mathcal{C}'$, then $F^{n+1}(\mathcal{C}') \subseteq F^n(\mathcal{C}')$ for all $n \geq 0$.*

Proof. The case $n = 0$ is true by assumption. For generic $n > 0$, we have that

$$\begin{aligned} F^{n+1}(\mathcal{C}') \subseteq F^n(\mathcal{C}') &\iff \forall (F^n c' \xrightarrow{F^n f} F^n c'') \in F^n(\mathcal{C}'),\, F(F^n c' \xrightarrow{F^n f} F^n c'') \in F^n(\mathcal{C}') \\ &\iff \forall (F^n c' \xrightarrow{F^n f} F^n c'') \in F^n(\mathcal{C}'),\, F^{n+1} c' \xrightarrow{F^{n+1} f} F^{n+1} c'' \in F^n(\mathcal{C}') \\ &\iff \forall (F^n c' \xrightarrow{F^n f} F^n c'') \in F^n(\mathcal{C}'),\, F^n(Fc') \xrightarrow{F^n(Ff)} F^n(Fc'')) \in F^n(\mathcal{C}') \end{aligned}$$

and the last condition holds by the case $n = 0$. ■

Lemma 5.8. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then for all integers $m \geq n$, $\mathcal{D}^{\leq -m} \subseteq \mathcal{D}^{\leq -n}$ and $\mathcal{D}^{\geq m} \subseteq \mathcal{D}^{\geq n}$.*

Proof. It suffices to consider $m = n+1$, from which the result follows by induction. By assumption, we have that

$$\mathcal{D}^{\leq -1} = \mathcal{D}^{\leq 0}[1] \subseteq \mathcal{D}^{\leq 0}, \quad \mathcal{D}^{\geq 1} = \mathcal{D}^{\geq 0}[-1] \subseteq \mathcal{D}^{\geq 0}.$$

Applying the above Lemma 5.7 then gives the desired result. ■

Lemma 5.9. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then for any $n \in \mathbb{Z}$, the pair $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$ is again a t-structure.*

Proof. To verify (T1), let $X \in \mathcal{D}^{\leq n}$ and $Y \in \mathcal{D}^{\geq n+1}$. Then

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X[n], Y[n]) = 0 \implies \mathrm{Hom}_{\mathcal{D}}(X, Y) = 0.$$

Axiom (T2) follows by the above Lemma 5.8. Finally, we check (T3). Let $X \in \mathcal{D}$. Then, by axiom (T3) for the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, we have $X' \in \mathcal{D}^{\leq 0}$, $X'' \in \mathcal{D}^{\geq 1}$ such that

$$X' \longrightarrow X[n] \longrightarrow X'' \longrightarrow X'[1]$$

is a distinguished triangle. By (TR2), we get a distinguished triangle

$$X'[-n] \longrightarrow X \longrightarrow X''[-n] \longrightarrow X'[-n+1]$$

and by definition $X'[-n] \in \mathcal{D}^{\leq n}$, $X''[-n] \in \mathcal{D}^{\geq n+1}$. ■

Theorem 5.10. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.*

(i) The inclusion $\iota_{\leq n} : \mathcal{D}^{\leq n} \to \mathcal{D}$ (resp. $\iota_{\geq n} : \mathcal{D}^{\geq 0} \to \mathcal{D}$) has a right (resp. left) adjoint $\tau^{\leq n}\colon \mathcal{D} \to \mathcal{D}^{\leq n}$ (resp. $\tau^{\geq n}\colon \mathcal{D} \to \mathcal{D}^{\geq n}$).

(ii) For all $n \in \mathbb{Z}$ and $X \in \mathcal{D}$, there exists a unique morphism $d_X^n \colon \tau^{\geq n+1}X \to (\tau^{\leq n}X)[1]$ such that

$$\tau^{\leq n}X \longrightarrow X \longrightarrow \tau^{\geq n+1}X \xrightarrow{d_X^n} (\tau^{\leq n}X)[1]$$

is a distinguished triangle. Furthermore, this data assembles into a natural transformation $d^n \colon \tau^{\geq n+1} \to [1] \circ \tau^{\leq n}$.

(iii) The unit η of the adjunction $(\tau^{\geq n+1}, \iota^{\geq n+1})$ and the counit ε of the adjunction $(\iota^{\leq n}, \tau^{\leq n})$ are given by the triangle

$$\tau^{\leq n}X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \tau^{\geq n+1}X \xrightarrow{d_X^n} (\tau^{\leq n}X)[1]$$

and hence any triangle of the form $X' \to X \to X'' \to X'[1]$ with $X' \in \mathcal{D}^{\leq n}$, $X'' \in \mathcal{D}^{\geq n+1}$ is canonically isomorphic to the one above.

Proof. (i) By the properties of the shift functor, we may assume $n = 0$. We will show that $\tau^{\leq 0}$ and $\tau^{\geq 1}$ exist. Let $Y \in \mathcal{D}$. To show the existence of these adjoint functors, we will show that the functors

$$F \colon \mathrm{Hom}_{\mathcal{D}}(\iota_{\leq 0}(-), Y) \colon \mathcal{D}^{\leq 0} \to \mathbf{Ab}, \quad G \colon \mathrm{Hom}_{\mathcal{D}}(Y, \iota_{\geq 1}(-)) \colon \mathcal{D}^{\geq 1} \to \mathbf{Ab}$$

are representable. By (T3) and (TR2), there exists $Y' \in \mathcal{D}^{\leq 0}$, $Y'' \in \mathcal{D}^{\geq 1}$ and a distinguished triangle

$$Y''[-1] \longrightarrow Y' \longrightarrow Y \longrightarrow Y''.$$

We show that $\mathrm{Hom}_{\mathcal{D}}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}^{\leq 0}}(X, Y')$ for all $X \in \mathcal{D}^{\leq 0}$. By Proposition 3.15, $\mathrm{Hom}_{\mathcal{D}}(X, -)$ is cohomological, and so from the above we get an exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(X, Y''[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y'')$$

in $\mathbf{Ab}$. Since $X \in \mathcal{D}^{\leq 0}$, by (T1) this reduces to

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y) \longrightarrow 0$$

from which we see that the map $\mathrm{Hom}_{\mathcal{D}^{\leq 0}}(X, Y') = \mathrm{Hom}_{\mathcal{D}}(X, Y') \to \mathrm{Hom}_{\mathcal{D}}(X, Y)$ induced by the distinguished triangle is an isomorphism. In particular, we obtain a natural isomorphism $F \cong \mathrm{Hom}(-, Y')$. An identical but dual computation applied to the distinguished triangle

$$Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow Y'[1]$$

shows that $G \cong \mathrm{Hom}(Y'', -)$. This establishes the existence of $\tau^{\leq 0}$ and $\tau^{\geq 1}$. We then have by standard abstract nonsense that $\tau^{\leq n}$, $\tau^{\geq n}$ exist and are given by

$$\tau^{\leq n} = [-n] \circ \tau^{\leq 0} \circ [n], \quad \tau^{\geq n} = [1 - n] \circ \tau^{\geq 1} \circ [n - 1].$$

(ii) When the map d_X^n exists, it is unique by Lemma 3.28. Lemma 5.9 implies that for each $X \in \mathcal{D}$, we can choose $X' \in \mathcal{D}^{\leq n}$, $X'' \in \mathcal{D}^{\geq n+1}$ such that

$$X' \xrightarrow{u} X \xrightarrow{v} X'' \xrightarrow{w} X'[1]$$

is a distinguished triangle. In (i), the way we show existence of $\tau^{\leq n}$, $\tau^{\geq n}$ actually shows that $\tau^{\leq n}X = X'$ and $\tau^{\geq n+1}X = X''$, and that the morphisms u, v are the canonical maps $\tau^{\leq n}X \to X$, $X \to \tau^{\geq n+1}X$. We conclude that we automatically have the desired distinguished triangle

$$\tau^{\leq n}X \longrightarrow X \longrightarrow \tau^{\geq n+1}X \xrightarrow{d_X^n} (\tau^{\leq n}X)[1].$$

The only thing which remains is to show that the d_X^n assemble into a natural transformation $\tau^{\geq n+1} \to [1] \circ \tau^{\leq n}$. Suppose we have a map $f : X \to Y$. Then, by (TR3) we obtain a map $\phi\colon \tau^{\geq n+1}X \to \tau^{\geq n+1}Y$ making the diagram

$$\begin{tikzcd}
\tau^{\leq n}X \arrow[r] \arrow[d, "\tau^{\leq n}f"] & X \arrow[r] \arrow[d, "f"] & \tau^{\geq n+1}X \arrow[r, "d_X^n"] \arrow[d, dashed, "\phi"] & (\tau^{\leq n}X)[1] \arrow[d] \\
\tau^{\leq n}Y \arrow[r] & Y \arrow[r] & \tau^{\geq n+1}Y \arrow[r, "d_Y^n"] & (\tau^{\leq n}Y)[1]
\end{tikzcd}$$

commute. The definition of adjointness immediately gives that the commutativity of the middle square, in particular the uniqueness of a map of the type of ϕ (see Proposition 3.25), implies that $\phi = \tau^{\geq n+1}f$.

(iii) This follows immediately from the construction in (i). ■

Definition 5.11. Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. The functors $\tau^{\leq n}\colon \mathcal{D} \to \mathcal{D}^{\leq n}$, $\tau^{\geq n}\colon \mathcal{D} \to \mathcal{D}^{\geq n}$ are called the *truncation functors.*

The uniqueness in Theorem 5.10 is quite useful, and allows us to deduce some more properties of the truncation functors.

Corollary 5.12. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.*

(i) If $X \in \mathcal{D}^{\leq n}$ (resp. $Y \in \mathcal{D}^{\geq n}$), then $\tau^{\leq n}X \to X$ (resp. $Y \to \tau^{\geq n}Y$) is an isomorphism.

(ii) If $X \in \mathcal{D}$, then $X \in \mathcal{D}^{\leq n}$ (resp. $X \in \mathcal{D}^{\geq n}$) if and only if $\tau^{\geq n+1}X = 0$ (resp. $\tau^{\leq n-1}X = 0$).

Proof. (i) This is a consequence of the fact that $\tau^{\geq n}$, $\tau^{\leq n}$ are adjoints of inclusions of full subcategories (see, for example, Lemma 3.77).

(ii) We have a distinguished triangle

$$\tau^{\leq n}X \longrightarrow X \longrightarrow \tau^{\geq n+1}X \longrightarrow (\tau^{\leq n}X)[1].$$

If $X \in \mathcal{D}^{\leq n}$, then (i) gives that the map $\tau^{\leq n}X \to X$ is an isomorphism, and we have morphisms

$$\begin{tikzcd}
\tau^{\leq n}X \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \tau^{\leq n}X \arrow[r] \arrow[d, "\wr"] & 0 \arrow[r] \arrow[d, dashed] & (\tau^{\leq n}X)[1] \arrow[d, equal] \\
\tau^{\leq n}X \arrow[r] & X \arrow[r] & \tau^{\geq n+1}X \arrow[r] & (\tau^{\leq n}X)[1]
\end{tikzcd}$$

which implies the dashed arrow is an isomorphism by Proposition 3.17. Conversely, if $\tau^{\geq n+1}X = 0$ then we have a distinguished triangle

$$\tau^{\leq n}X \longrightarrow X \longrightarrow 0 \longrightarrow (\tau^{\leq n}X)[1]$$

which by Lemma 3.18 gives that $\tau^{\leq n}X \xrightarrow{\sim} X$. Since t-structures are closed under isomorphism, this implies $X \in \mathcal{D}^{\leq n}$. The remaining things follow by a dual proof. ■

In the definition of a t-structure, we specify two full subcategories. This is actually not necessary: it suffices to know just $\mathcal{D}^{\leq 0}$ (or $\mathcal{D}^{\geq 0}$). We can obtain this as a corollary of the preceding corollary.

Lemma 5.13. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then $0 \in \mathcal{D}^{\leq n}$ and $0 \in \mathcal{D}^{\geq n}$ for all $n \in \mathbb{Z}$. In particular, $0 \in \mathcal{D}^{\heartsuit}$.*

Proof. Since $\tau^{\leq n}$ is a right adjoint, it preserves limits. Therefore, $\tau^{\leq n}0$ is initial, hence by Proposition 2.6, it is terminal and therefore $\tau^{\leq n}0 = 0$. Similarly, $\tau^{\geq n}$ is a left adjoint, so it preserves colimits. Therefore $\tau^{\geq 0}0$ is terminal, and so by Proposition 2.6 it is initial, hence $\tau^{\geq n}0 = 0$. ∎

Corollary 5.14. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then*

(i) $Y \in \mathcal{D}^{\geq 1}$ if and only if for all $X \in \mathcal{D}^{\leq 0}$, $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$, and

(ii) $X \in \mathcal{D}^{\leq 0}$ if and only if for all $Y \in \mathcal{D}^{\geq 1}$, $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$.

Proof. We prove (i), since (ii) follows by a dual proof. When $Y \in \mathcal{D}^{\geq 1}$, $\mathrm{Hom}(X, Y) = 0$ by definition of a t-structure. Conversely, suppose $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in \mathcal{D}^{\leq 0}$. Then

$$0 = \mathrm{Hom}(X, 0) = \mathrm{Hom}(X, Y) \cong \mathrm{Hom}(X, \tau^{\leq 0}Y)$$

so $\tau^{\leq 0}Y = 0$. Therefore, $Y \in \mathcal{D}^{\geq 1}$. ∎

Remark 5.15. Let $\mathcal{D}$ be an arbitrary triangulated category, and let $\mathcal{C} \subseteq \mathcal{D}$ be a full subcategory. We can think of $\mathrm{Hom}_{\mathcal{D}}(-,-)$ as being analogous to an inner product in linear algebra. This heuristic allows us to denote by $\mathcal{C}^{\perp}$ and ${}^{\perp}\mathcal{C}$ the full subcategories defined by

$$\mathcal{C}^{\perp} = \{Y \in \mathcal{D} \mid \forall X \in \mathcal{C}, \mathrm{Hom}(X, Y) = 0\}, \quad {}^{\perp}\mathcal{C} = \{X \in \mathcal{D} \mid \forall Y \in \mathcal{C}, \mathrm{Hom}(X, Y) = 0\}.$$

Observe then that if $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure, the above corollary states that $(\mathcal{D}^{\leq 0})^{\perp} = \mathcal{D}^{\geq 1}$ and ${}^{\perp}(\mathcal{D}^{\geq 1}) = \mathcal{D}^{\leq 0}$. That is, these are "orthogonal complements" of each other.

Corollary 5.16. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then the full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ are closed under direct summands. That is, if $X \oplus X' \in \mathcal{D}^{\leq 0}$ (resp. $X \oplus X' \in \mathcal{D}^{\geq 0}$), then $X, X' \in \mathcal{D}^{\leq 0}$ (resp. $X, X' \in \mathcal{D}^{\geq 0}$).*

Proof. We show only one assertion, since the other is dual. Let $Y \in \mathcal{D}^{\geq 1}$. If $X \oplus X' \in \mathcal{D}^{\leq 0}$, we have

$$0 = \mathrm{Hom}(X \oplus X', Y) \cong \mathrm{Hom}(X, Y) \times \mathrm{Hom}(X', Y) \implies \mathrm{Hom}(X, Y) = \mathrm{Hom}(X', Y) = 0$$

and therefore $X, X' \in {}^{\perp}(\mathcal{D}^{\geq 1}) = \mathcal{D}^{\leq 0}$. ∎

The above Corollary 5.16 allows us to prove a neat lemma from which we can obtain yet another consequence of Corollary 5.12.

Lemma 5.17. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, and let $X \in \mathcal{D}$. If $\mathrm{Hom}(X, \tau^{\geq 1}X) = 0$, then $X \in \mathcal{D}^{\leq 0}$. Similarly, if $\mathrm{Hom}(\tau^{\leq -1}X, X) = 0$ then $X \in \mathcal{D}^{\geq 0}$.*

Proof. We prove the first assertion, since the other is dual. We have the distinguished triangle

$$\tau^{\leq 0}X \longrightarrow X \xrightarrow{0} \tau^{\geq 1}X \longrightarrow (\tau^{\leq 0}X)[1]$$

which in particular implies we have an isomorphism of triangles

$$\begin{tikzcd}
(\tau^{\geq 1}X)[-1] \arrow[r] \arrow[d, equal] & \tau^{\leq 0}X \arrow[r] \arrow[d, "\wr"] & X \arrow[r, "0"] \arrow[d, equal] & \tau^{\geq 1}X \arrow[d, equal] \\
(\tau^{\geq 1}X)[-1] \arrow[r] & (\tau^{\geq 1}X)[-1] \oplus X \arrow[r] & X \arrow[r, "0"] & \tau^{\geq 1}X
\end{tikzcd}$$

so that X is a direct summand of $\tau^{\leq 0}X$. By Corollary 5.16, $\mathcal{D}^{\leq 0}$ is closed under direct summands, so $X \in \mathcal{D}^{\leq 0}$. ■

The corollary we want from this is as follows:

Corollary 5.18. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. If in a distinguished triangle*

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$$

we have $X', X'' \in \mathcal{D}^{\leq 0}$ (resp. $X', X'' \in \mathcal{D}^{\geq 0}$) then $X \in \mathcal{D}^{\leq 0}$ (resp. $X \in \mathcal{D}^{\geq 0}$).

Proof. We prove this for the case $X', X'' \in \mathcal{D}^{\leq 0}$. The other case follows by a dual argument. Since $X', X'' \in \mathcal{D}^{\leq 0}$, we have that $\mathrm{Hom}(X', \tau^{\geq 1}X) = \mathrm{Hom}(X'', \tau^{\geq 1}X) = 0$ by (T1) and the definition of $\tau^{\geq 1}X$. Since $\mathrm{Hom}(-, \tau^{\geq 1}X)$ is cohomological, we have the exact sequence

$$0 = \mathrm{Hom}(X'', \tau^{\geq 1}X) \longrightarrow \mathrm{Hom}(X, \tau^{\geq 1}X) \longrightarrow \mathrm{Hom}(X', \tau^{\geq 1}X) = 0$$

and so $\mathrm{Hom}(X, \tau^{\geq 1}X) = 0$. This implies that $X \in \mathcal{D}^{\leq 0}$ by Lemma 5.17. ■

5.2 The Heart is Abelian

We return to the example case of the derived category $\mathbf{D}(\mathcal{A})$ of an Abelian category. We have a functor $\mathcal{A} \to \mathbf{D}(\mathcal{A})$ given by the composition

$$\mathcal{A} \longrightarrow \mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A}),$$

i.e. by sending $A \in \mathcal{A}$ to the complex concentrated in degree zero. Now consider the standard t-structure $(\mathbf{D}(\mathcal{A})^{\leq 0}, \mathbf{D}(\mathcal{A})^{\geq 0})$ on $\mathbf{D}(\mathcal{A})$. The heart $\mathbf{D}(\mathcal{A})^{\heartsuit}$ of this t-structure consists of the full subcategory spanned by complexes concentrated in degree zero. By Theorem 4.41, we then know that the above functor induces an equivalence between $\mathbf{D}(\mathcal{A})^{\heartsuit}$ and $\mathcal{A}$. In particular, the heart is an Abelian category. Remarkably, this is always the case in any triangulated category with a t-structure.

Proposition 5.19. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then the heart $\mathcal{D}^{\heartsuit}$ is a full additive subcategory of $\mathcal{D}$.*

Proof. By Lemma 5.13, $0 \in \mathcal{D}^{\heartsuit}$. To show that $\mathcal{D}^{\heartsuit}$ has direct sums, we use the orthogonality condition from Corollary 5.14. In particular, let $Z \in \mathcal{D}^{\leq -1}$. Then

$$\mathrm{Hom}(Z, X \oplus Y) \cong \mathrm{Hom}(Z, X) \oplus \mathrm{Hom}(Z, Y) = 0$$

since $X, Y \in \mathcal{D}^{\geq 0}$. This shows that $X \oplus Y \in \mathcal{D}^{\geq 0}$. Similarly, let $W \in \mathcal{D}^{\geq 1}$. Then

$$\mathrm{Hom}(X \oplus Y, W) \cong \mathrm{Hom}(X, W) \oplus \mathrm{Hom}(Y, W) = 0$$

since $X, Y \in \mathcal{D}^{\leq 0}$. This shows that $X \oplus Y \in \mathcal{D}^{\leq 0}$. Therefore, $X \in \mathcal{D}^{\heartsuit}$.

Since $\mathcal{D}^{\heartsuit}$ is a full subcategory of an additive category, it is pre-additive. Since it admits a zero object and finite direct sums, it is additive. ■

Theorem 5.20. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then $\mathcal{D}^{\heartsuit}$ is an Abelian category. Furthermore, if $f\colon X \to Y$ is a morphism in $\mathcal{D}^{\heartsuit}$ and $X \to Y \to Z \to X[1]$ is a distinguished triangle extending this, then $\tau^{\geq 0} Z = \operatorname{coker} f$ and $\tau^{\leq 0}(Z[-1]) = \ker f$ in $\mathcal{D}^{\heartsuit}$.*

Proof. By Proposition 5.19, $\mathcal{D}^{\heartsuit}$ is additive. Hence, what remains is to construct the kernel and cokernel, and to show that the image is the coimage.

Let $f\colon X \to Y$ be a morphism in $\mathcal{D}^{\heartsuit}$. Then we have a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

and shifting this to the left shows that $Z \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$. Intuitively, Z lives in degrees -1 and 0, and the idea is that we will try to show that the two parts it contains are precisely the kernel and cokernel. For all $W \in \mathcal{D}^{\heartsuit}$, we have exact sequences

$$\operatorname{Hom}(X[1], W) \longrightarrow \operatorname{Hom}(Z, W) \longrightarrow \operatorname{Hom}(Y, W) \longrightarrow \operatorname{Hom}(X, W),$$

$$\operatorname{Hom}(W, Y[-1]) \longrightarrow \operatorname{Hom}(W, Z[-1]) \longrightarrow \operatorname{Hom}(W, X) \longrightarrow \operatorname{Hom}(W, Y).$$

Since $W \in \mathcal{D}^{\heartsuit}$, we see that $\operatorname{Hom}(W, Y[-1]) = 0$ and $\operatorname{Hom}(X[1], W) = 0$. Furthermore, we have natural isomorphisms

$$\operatorname{Hom}(Z, W) \cong \operatorname{Hom}(\tau^{\geq 0} Z, W) \quad \text{and} \quad \operatorname{Hom}(W, Z[-1]) \cong \operatorname{Hom}(W, \tau^{\leq 0}(Z[-1])).$$

Thus, in totality, we have the exact sequences

$$0 \longrightarrow \operatorname{Hom}(\tau^{\geq 0} Z, W) \longrightarrow \operatorname{Hom}(Y, W) \longrightarrow \operatorname{Hom}(X, W),$$

$$0 \longrightarrow \operatorname{Hom}(W, \tau^{\leq 0}(Z[-1])) \longrightarrow \operatorname{Hom}(W, X) \longrightarrow \operatorname{Hom}(W, Y).$$

By the Yoneda lemma, this implies that $\tau^{\geq 0} Z = \operatorname{coker} f$ and $\tau^{\leq 0}(Z[-1]) = \ker f$.

We have thus established that $\mathcal{D}^{\heartsuit}$ is additive and admits kernels and cokernels. It remains to check that the image is the coimage. Place the map $Y \to \tau^{\geq 0} Z = \operatorname{coker} f$ into a distinguished triangle

$$Y \longrightarrow \tau^{\geq 0} Z \longrightarrow E \longrightarrow Y[1]$$

and shift to the right to obtain the distinguished triangle

$$E[-1] \longrightarrow Y \longrightarrow \tau^{\geq 0} Z \longrightarrow E.$$

Note that by shifting the first triangle, we see that $E \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$. Since E "contains" both the kernel and cokernel of the map $Y \to \tau^{\geq 0} Z$, we will write $I = E[-1]$ (where we see that $I \in \mathcal{D}^{\leq 1} \cap \mathcal{D}^{\geq 0}$) so that $\tau^{\leq 0} I$ is the image of f. We now consider the three distinguished triangles

$$\left\{ \begin{array}{ccccccc} Y & \longrightarrow & Z & \longrightarrow & X[1] & \longrightarrow & Y[1] \\ Y & \longrightarrow & \tau^{\geq 0} Z & \longrightarrow & I[1] & \longrightarrow & Y[1] \\ Z & \longrightarrow & \tau^{\geq 0} Z & \longrightarrow & (\tau^{\leq -1} Z)[1] & \longrightarrow & Z[1] \end{array} \right.$$

and apply (TR4) to obtain the commutative diagram

$$\begin{tikzcd}
{} & {} & {} & {} & {} & {} \\
{} & {} & X[1] \arrow[ur] \arrow[dr, dashed] & {} & {} & {} \\
{} & Z \arrow[ur] \arrow[dr] & {} & I[1] \arrow[ur] \arrow[ddr, dashed] & {} & {} \\
{} & {} & \tau^{\geq 0}Z \arrow[ur] \arrow[drr] & {} & {} & {} \\
Y \arrow[uur] \arrow[urr] & {} & {} & {} & (\tau^{\leq -1}Z)[1] \arrow[r] \arrow[dr, dashed] & {} \\
{} & {} & {} & {} & {} & {}
\end{tikzcd}$$

from which, after shifting, we extract a distinguished triangle

$$(\tau^{\leq -1}Z)[-1] \longrightarrow X \longrightarrow I \longrightarrow \tau^{\leq -1}Z.$$

By shifting to the left, we see that $I \in \mathcal{D}^{\leq 0}$, so $I \in \mathcal{D}^{\heartsuit}$. We have that

$$[-1] \circ \tau^{\leq -1} = [-1] \circ [1] \circ \tau^{\leq 0} \circ [1] \implies (\tau^{\leq -1}Z)[-1] = \tau^{\leq 0}(Z[-1]) = \ker f,$$

so we really have a distinguished triangle

$$\ker f \longrightarrow X \longrightarrow I \longrightarrow \tau^{\leq -1}Z.$$

which shows that $I = \tau^{\geq 0}I = \operatorname{coim} f$ (where the first equality is from $I \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$). Similarly, the shift

$$Y \longrightarrow \operatorname{coker} f \longrightarrow I[1] \longrightarrow Y[1]$$

of the distinguished triangle that we started with shows that $\operatorname{im} f = \tau^{\leq 0}I = I$. Thus, $\operatorname{coim} f = \operatorname{im} f$, and we have shown that $\mathcal{D}^{\heartsuit}$ is an Abelian category. ■

5.3 COHOMOLOGY FUNCTORS & MORE ON TRUNCATION

Recall again that $\mathbf{D}(\mathcal{A})^{\heartsuit} \cong \mathcal{A}$ with the standard t-structure. This implies that for all n, the cohomology functor $\mathrm{H}^n \colon \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ can be made into a functor $\mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})^{\heartsuit}$. Furthermore, note that it is easily seen that the cohomology functor is given by $\tau^{\leq n}\tau^{\geq n}$ (which is also clearly equivalent to $\tau^{\leq n}\tau^{\geq n}$ in this situation). We can replicate this in the generality of t-structures.

Definition 5.21. Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. The *zeroth cohomology functor* with respect to this t-structure is

$$\mathrm{H}^0 \colon \mathcal{D} \to \mathcal{D}^{\heartsuit}, \quad \mathrm{H}^0 := \tau^{\geq 0} \circ \tau^{\leq 0}.$$

More generally, for any $n \in \mathbb{Z}$ we define the nth cohomology functor

$$\mathrm{H}^n \colon \mathcal{D} \to \mathcal{D}^{\heartsuit}, \quad \mathrm{H}^n := \mathrm{H}^0 \circ [n]$$

Remark 5.22. Note that since

$$\tau^{\geq n} \circ \tau^{\leq n} = [-n] \circ \tau^{\geq 0} \circ [n] \circ [-n] \circ \tau^{\leq 0} \circ [n] = [-n] \circ \tau^{\geq 0} \circ \tau^{\leq 0} \circ [n]$$

we have

$$\mathrm{H}^n = \tau^{\geq 0} \circ \tau^{\leq 0} \circ [n] = [n] \circ ([-n] \circ \tau^{\geq 0} \circ \tau^{\leq 0} \circ [n]) = [n] \circ \tau^{\geq n} \circ \tau^{\leq n}.$$

Before we say anything about these cohomology functors, we should check that the truncation functors interact well with each other. In particular, above we chose to write $\mathrm{H}^0 = \tau^{\geq 0} \circ \tau^{\leq 0}$, but we could've chosen the opposite composition too. This would still have given a well-defined functor to the heart. We really want these two to agree.

Proposition 5.23. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then*

(i) if $m \leq n$, we have natural isomorphisms

$$\tau^{\leq n} \circ \tau^{\leq m} \cong \tau^{\leq m} \circ \tau^{\leq n} \cong \tau^{\leq m} \quad \text{and} \quad \tau^{\geq n} \circ \tau^{\geq m} \cong \tau^{\geq m} \circ \tau^{\geq n} \cong \tau^{\geq n},$$

(ii) if $m > n$, then $\tau^{\geq m} \circ \tau^{\leq n} = \tau^{\leq n} \circ \tau^{\geq m} = 0$, and

(iii) for all $m, n \in \mathbb{Z}$, $\tau^{\leq n} \circ \tau^{\geq m} \cong \tau^{\geq m} \circ \tau^{\leq n}$, and for each $X \in \mathcal{D}$ there is a unique map $\tau^{\geq m}\tau^{\leq n}X \to \tau^{\leq n}\tau^{\geq m}X$ such that

$$\begin{tikzcd}
\tau^{\leq n}X \arrow[r] \arrow[d] & X \arrow[r] & \tau^{\geq m}X \\
\tau^{\geq m}\tau^{\leq n}X \arrow[rr] & & \tau^{\leq n}\tau^{\geq m}X \arrow[u]
\end{tikzcd}$$

commutes which is furthermore an isomorphism.

Proof. (i) Let $X \in \mathcal{D}$. Since $\tau^{\leq m}X \in \mathcal{D}^{\leq m}$, we have that the canonical map $\tau^{\leq n}\tau^{\leq m}X \to \tau^{\leq m}X$ is an isomorphism by Corollary 5.12, which implies we have an equivalence $\tau^{\leq n} \circ \tau^{\leq m} = \tau^{\leq m}$. To see that $\tau^{\leq m} \circ \tau^{\leq n} = \tau^{\leq m}$, take $Y \in \mathcal{D}^{\leq m} \subseteq \mathcal{D}^{\leq n}$ and note that we have natural isomorphisms

$$\mathrm{Hom}(Y, \tau^{\leq m}\tau^{\leq n}X) \cong \mathrm{Hom}(Y, \tau^{\leq n}X) \cong \mathrm{Hom}(Y, X) \cong \mathrm{Hom}(Y, \tau^{\leq m}X)$$

so $\mathrm{Hom}(-, \tau^{\leq m}\tau^{\leq n}X) = \mathrm{Hom}(-, \tau^{\leq m}X)$. This gives the equivalence $\tau^{\leq m} \circ \tau^{\leq n} = \tau^{\leq m}$. The statements for $\tau^{\geq *}$ follow similarly from a dual argument.

(ii) Let $X \in \mathcal{D}$. Then $\tau^{\geq m}X \in \mathcal{D}^{\geq m}$, so Corollary 5.12 together with part (i) says that $\tau^{\leq n}\tau^{\geq m}X = 0$ since $n < m$. Similarly, $\tau^{\leq n}X \in \mathcal{D}^{\leq n}$, so $\tau^{\geq m}\tau^{\leq n}X = 0$.

(iii) By part (ii), we may assume that $m \leq n$. By Theorem 5.10(ii), we have the two distinguished triangles

$$\tau^{\leq n}\tau^{\geq m}X \longrightarrow \tau^{\geq m}X \longrightarrow \tau^{\geq n+1}\tau^{\geq m}X \longrightarrow (\tau^{\leq n}\tau^{\geq m}X)[1]$$

$$\tau^{\leq m-1}\tau^{\leq n}X \longrightarrow \tau^{\leq n}X \longrightarrow \tau^{\geq m}\tau^{\leq n}X \longrightarrow (\tau^{\leq m-1}\tau^{\leq n}X)[1]$$

which, using part (i) and (TR2) we can rewrite as

$$(\tau^{\geq n+1}X)[-1] \longrightarrow \tau^{\leq n}\tau^{\geq m}X \longrightarrow \tau^{\geq m}X \longrightarrow \tau^{\geq n+1}X$$

$$\tau^{\leq n}X \longrightarrow \tau^{\geq m}\tau^{\leq n}X \longrightarrow (\tau^{\leq m-1}X)[1] \longrightarrow (\tau^{\leq n}X)[1]$$

and Corollary 5.18 then tells us that $\tau^{\leq n}\tau^{\geq m}X, \tau^{\geq m}\tau^{\leq n}X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$. We then have natural isomorphisms

$$\mathrm{Hom}(\tau^{\geq m}\tau^{\leq n}X, \tau^{\leq n}\tau^{\geq m}X) \cong \mathrm{Hom}(\tau^{\geq m}\tau^{\leq n}X, \tau^{\geq m}X) \cong \mathrm{Hom}(\tau^{\leq n}X, \tau^{\geq m}X),$$

which allows us to produce a unique map $\tau^{\geq m}\tau^{\leq n}X \to \tau^{\leq n}\tau^{\geq m}X$ from the composition $\tau^{\leq n}X \to X \to \tau^{\geq m}X$, which by the nature of the above canonical isomorphisms implies exactly that the desired diagram commutes.

Let $\phi\colon \tau^{\geq m}\tau^{\leq n}X \to \tau^{\leq n}\tau^{\geq m}X$ be the morphism constructed above. We now wish to show that it is an isomorphism. First, we use the octahedral axiom (TR4) on the distinguished triangles

$$\left\{\begin{array}{ccccccc} \tau^{\leq m-1}X & \longrightarrow & \tau^{\leq n}X & \longrightarrow & \tau^{\geq m}\tau^{\leq n}X & \longrightarrow & (\tau^{\leq m-1}X)[1] \\ \tau^{\leq m-1}X & \longrightarrow & X & \longrightarrow & \tau^{\geq m}X & \longrightarrow & (\tau^{\leq m-1}X)[1] \\ \tau^{\leq n}X & \longrightarrow & X & \longrightarrow & \tau^{\geq n+1}X & \longrightarrow & (\tau^{\leq n}X)[1] \end{array}\right.$$

to obtain a commutative diagram

$$\begin{tikzcd}
 & & & {} & & \\
 & & \tau^{\geq m}\tau^{\leq n}X \arrow[ur] \arrow[dr, dashed, "f"] & & {} & \\
 & \tau^{\leq n}X \arrow[ur] \arrow[dr] & & \tau^{\geq m}X \arrow[ur] \arrow[ddr, dashed, "g"] & & \\
 & & X \arrow[ur] \arrow[drr] & & & \\
\tau^{\leq m-1}X \arrow[uur] \arrow[urr] & & & & \tau^{\geq n+1}X \arrow[dr] \arrow[ddr, dashed] & \\
 & & & & & {} \\
 & & & & & {}
\end{tikzcd}$$

from which we get the distinguished triangle

$$\tau^{\geq m}\tau^{\leq n}X \xrightarrow{f} \tau^{\geq m}X \xrightarrow{g} \tau^{\geq n+1}X \longrightarrow (\tau^{\geq m}\tau^{\leq n}X)[1].$$

However, since $\tau^{\geq m}\tau^{\leq n}X \in \mathcal{D}^{\leq n}\cap\mathcal{D}^{\geq m}$ and $\tau^{\geq n+1}X \in \mathcal{D}^{\geq n+1}$, and the fact that a representation of this type is unique (which we see from the proof of Theorem 5.10, essentially by the uniqueness of adjoints), we get that $\tau^{\geq m}\tau^{\leq n}X = \tau^{\leq n}\tau^{\geq m}X$. ■

As a result of the above proposition, we have natural isomorphisms

$$\mathrm{H}^0 \cong \tau^{\geq 0} \circ \tau^{\leq 0} \cong \tau^{\leq 0} \circ \tau^{\geq 0}$$

so it doesn't matter which order we truncate.

Corollary 5.24. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then for any map $f\colon X \to Y$ in the heart $\mathcal{D}^{\heartsuit}$ which extends to a distinguished triangle $X \to Y \to Z \to X[1]$, we have that*

$$\ker f = \mathrm{H}^{-1}(Z) \quad \textit{and} \quad \operatorname{coker} f = \mathrm{H}^0(Z).$$

Proof. This is an immediate consequence of Theorem 5.20 together with Proposition 5.23. ■

Remark 5.25. In the notation of Remark 3.3, we see that $\ker f \cong \mathrm{H}^{-1}(C_f) \cong \mathrm{H}^0(K_f)$ and $\operatorname{coker} f \cong \mathrm{H}^0(C_f) \cong \mathrm{H}^1(K_f)$.

We can use the cohomology functors to detect, in some cases, when an object is in $\mathcal{D}^{\leq 0}$ or $\mathcal{D}^{\geq 0}$.

Proposition 5.26. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $X \in \mathcal{D}$, and suppose that there exists some $n \in \mathbb{Z}$ such that $X \in \mathcal{D}^{\leq n}$ (resp. $X \in \mathcal{D}^{\geq n}$). Then $X \in \mathcal{D}^{\leq 0}$ (resp. $X \in \mathcal{D}^{\geq 0}$) if and only if $\mathrm{H}^i(X) = 0$ for all $i > 0$ (resp. $i < 0$).*

Proof. We consider the case $X \in \mathcal{D}^{\leq n}$, since the other one follows by a dual proof. The map $\tau^{\leq n}X \to X$ is an isomorphism since $X \in \mathcal{D}^{\leq n}$. If $X \in \mathcal{D}^{\leq 0}$, then it is clear that $\mathrm{H}^i(X) = 0$ for all $i > 0$ by Corollary 5.12. Conversely, if $X \in \mathcal{D}^{\leq n}$ and $n \leq 0$, then we already know $X \in \mathcal{D}^{\leq n} \subseteq \mathcal{D}^{\leq 0}$, so all statements in the proposition are verified. In the other case, i.e. $n > 0$, we see that if $\mathrm{H}^i(X) = 0$ for all $i > 0$, then in particular $\mathrm{H}^n(X) = 0$. We then have a distinguished triangle

$$\tau^{\leq n-1}\tau^{\leq n}X \longrightarrow \tau^{\leq n}X \longrightarrow \tau^{\geq n}\tau^{\leq n}X \longrightarrow (\tau^{\leq n-1}\tau^{\leq n}X)[1]$$

which we compute as

$$\tau^{\leq n-1}X \longrightarrow X \longrightarrow \mathrm{H}^n(X)[-n] \longrightarrow (\tau^{\leq n-1}X)[1]$$

from which we obtain the distinguished triangle

$$\tau^{\leq n-1}X \longrightarrow X \longrightarrow 0 \longrightarrow (\tau^{\leq n-1}X)[1].$$

By Lemma 3.18, the map $\tau^{\leq n-1}X \to X$ is an isomorphism, and therefore $X \in \mathcal{D}^{\leq n-1}$. Continuing by induction shows that $X \in \mathcal{D}^{\leq 0}$. ■

Corollary 5.27. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then if*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is a short exact sequence in $\mathcal{D}^{\heartsuit}$, there is a unique map $Z \to X[1]$ such that

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

is a distinguished triangle.

Proof. We begin by extending the first map into a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z' \longrightarrow X[1].$$

This triangle tells us that $Z' \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$. By the first isomorphism theorem (Theorem 2.28), exactness says that $\mathrm{H}^{-1}(Z') = \ker(X \to Y) = \mathrm{im}(0 \to X) = 0$, and $\mathrm{H}^0(Z') = \mathrm{coker}(X \to Y) = Z$. We therefore know that $\mathrm{H}^i(Z') = 0$ for all $i < 0$, so $Z' \in \mathcal{D}^{\geq 0}$, which implies that $Z' \in \mathcal{D}^{\heartsuit}$. Thus $Z' = \mathrm{H}^0(Z') = Z$. We therefore obtain the desired map $Z \to X[1]$ from the map $Z' \to X[1]$. Uniqueness follows from Lemma 3.28. ■

5.4 Extensions in the Heart

In an Abelian category $\mathcal{A}$ with enough injectives, it is possible to define the Ext-groups $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$ for any $A, B \in \mathcal{A}$ using standard homological methods (i.e. by taking an injective resolution of B). In the situation of a triangulated category $\mathcal{D}$ with a t-structure, we do not know that the heart $\mathcal{D}^{\heartsuit}$ is nice enough to have enough injectives. In such a circumstance, it is still possible to define Ext-groups without utilizing injective (or projective) resolutions, as was done by Yoneda in [Yon60]. A more modern reference would be [MLan95].

We will take various facts about these Yoneda Ext-groups as known to the reader in the following exposition. In particular, we will make the definition

Definition 5.28. Let $\mathcal{A}$ be an Abelian category, and let A, B in $\mathcal{A}$. We let $\mathrm{Ext}(A, B)$ be the Abelian group whoose elements are extensions

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

up to Yoneda equivalence (*congruence* in [MLan95]) and whose addition is given by the Baer sum, i.e. $\mathcal{E} + \mathcal{E}' = (\nabla_B)_*(\Delta_A)^*(\mathcal{E} \oplus \mathcal{E}')$ for $\mathcal{E}, \mathcal{E}' \in \mathrm{Ext}(A, B)$.

Facts about Ext (e.g. functoriality, the fact that it is well-defined at all, facts about the Baer sum, and so forth) are assumed as prerequisites. We now provide a characterization of Ext when $\mathcal{A} = \mathcal{D}^\heartsuit$.

Remark 5.29. One should note that $\operatorname{Ext}(A, B)$ need not be a small set.

Theorem 5.30. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $X, Y \in \mathcal{D}^\heartsuit$. Then* $\operatorname{Hom}(X, Y[1]) = \operatorname{Ext}(X, Y)$*. In particular, we have a natural isomorphism*

$$\operatorname{Hom}_{\mathcal{D}^\heartsuit}(-, (-)[1]) \cong \operatorname{Ext}(-, -).$$

Proof. We construct a map $\operatorname{Hom}(X, Y[1]) \to \operatorname{Ext}(X, Y)$ as follows: consider a morphism $f : X \to Y[1]$. Then by (TR1) we can complete this to a distinguished triangle

$$X \longrightarrow Y[1] \longrightarrow Z \longrightarrow X[1]$$

which, after shifting using (TR2), becomes

$$Y \xrightarrow{\ i\ } E \xrightarrow{\ p\ } X \xrightarrow{\ f\ } Y[1]$$

where $E := Z[-1]$. This is our prospective element of $\operatorname{Ext}(X, Y)$. In particular, we must show that the sequence

$$0 \longrightarrow Y \xrightarrow{\ i\ } E \xrightarrow{\ p\ } X \longrightarrow 0$$

is exact. To prove this, we will first unravel the distinguished triangle to

$$X[-1] \xrightarrow{-f[-1]} Y \xrightarrow{\ i\ } E \xrightarrow{\ p\ } X \xrightarrow{\ f\ } Y[1].$$

Let $W \in \mathcal{D}^\heartsuit$. Since $\operatorname{Hom}(W, -)$ is cohomological, we obtain an exact sequence

$$0 \longrightarrow \operatorname{Hom}(W, Y) \xrightarrow{\ i\,\circ\ } \operatorname{Hom}(W, E) \xrightarrow{\ p\,\circ\ } \operatorname{Hom}(X, W).$$

From this, we get that composition with i on the left is injective so that i is a monomorphism, which implies exactness at Y. Furthermore, since i is a monomorphism, we have that $\operatorname{im} i = Y$. Therefore, to check exactness at E we need to check that $\ker p = Y$. This follows from the above exact sequence, in particular exactness at $\operatorname{Hom}(W, E)$: suppose we have some map $g \colon W \to E$ such that $p \circ g = 0$. Then $g \in \ker(p\,\circ)$ and therefore exactness tells us that there is some $h \colon W \to Y$ such that $g = i \circ h$. Furthermore, injectivity of left composition with i implies that this choice is unique, and therefore Y satisfies the universal property of the kernel.

To check exactness at X, we consider the cohomological functor $\operatorname{Hom}(-, W)$. From this, we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}(X, W) \xrightarrow{\ \circ\, p\ } \operatorname{Hom}(E, W) \xrightarrow{\ \circ\, i\ } \operatorname{Hom}(Y, W).$$

Hence right composition with p is injective, so p is an epimorphism. We note that $\operatorname{im} p = X$ if and only if $\operatorname{coker} i = X$. In particular,

$$\operatorname{im} p = \operatorname{coker}(\ker p \to E) \cong \operatorname{coker}(\operatorname{im} i \to E) = \operatorname{coker} i.$$

So we need to show that $\operatorname{coker} i = X$. However, this follows from the exactness of the above diagram: suppose we have a map $g \colon E \to W$ such that $g \circ i = 0$, i.e. $g \in \ker(\circ\, i)$. Then exactness says that we have some $h \colon X \to W$ such that $g = h \circ p$, and the choice is unique since p is an epimorphism. Therefore, X is the cokernel of i, so we have exactness at X.

Thus, from the morphism $f : X \to Y[1]$, we have constructed an exact sequence. This prospectively defines a map of sets $\mathrm{Hom}(X, Y[1]) \to \mathrm{Ext}(X, Y)$, with the caveat that we have to check that it is well-defined (i.e. independent of the choice of cone), but this follows trivially using Proposition 3.17. We must now check that this is natural, bijective, and compatible with the group structures involved. We start by showing that it is natural.

Fix $Y \in \mathcal{D}^{\heartsuit}$, and consider the functors $\mathrm{Hom}(-, Y[1])$ and $\mathrm{Ext}(-, Y)$. Let $\rho_{(-)}\colon \mathrm{Hom}(-, Y[1]) \to \mathrm{Ext}(-, Y)$ denote the map defined above. We must show that, for any morphism $\alpha\colon X \to X'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}(X, Y[1]) \arrow[r, "\rho_X"] & \mathrm{Ext}(X, Y) \\
\mathrm{Hom}(X', Y[1]) \arrow[r, "\rho_{X'}"] \arrow[u, "\alpha^*"] & \mathrm{Ext}(X', Y) \arrow[u, "\alpha^*"']
\end{tikzcd}$$

commutes. Let $f \in \mathrm{Hom}(X', Y[1])$. Then the extensions $\rho_{X'}(f)$, $\alpha^*\rho_{X'}(f)$, and $\rho_X(\alpha^* f)$ are the top, middle, and bottom row of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & K_f \arrow[r] & X' \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & K_f \times_{X'} X \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "\alpha"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & K_{f\circ\alpha} \arrow[r] \arrow[u, dotted, "?"] & X \arrow[r] & 0
\end{tikzcd} \tag{5}$$

We then see that if we can produce a map $K_{f\circ\alpha} \to K_f$ which we can apply the universal property of $K_f \times_{X'} X$ to, we will produce a map filling in the arrow indicated by "?" in the above diagram. Then we will see that $\rho_X(\alpha^* f) = \alpha^*\rho_{X'}(f)$. Actually, we also need to check that the bottom left square commutes, but from the way we construct the map $K_{f\circ\alpha} \to K_f$ this will follow by comparing components. We will construct the desired map using (TR4). In particular, by applying it to α, f, and $f \circ \alpha$, i.e. the triangles

$$\left\{ \begin{array}{ccccccc}
X & \xrightarrow{\alpha} & X' & \longrightarrow & K_\alpha[1] & \longrightarrow & X[1] \\
X & \xrightarrow{f\circ\alpha} & Y[1] & \longrightarrow & K_{f\circ\alpha}[1] & \longrightarrow & X[1] \\
X' & \xrightarrow{f} & Y[1] & \longrightarrow & K_f[1] & \longrightarrow & X'[1]
\end{array} \right.$$

we obtain the diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, equal] & X' \arrow[r] \arrow[d, "f"] & K_\alpha[1] \arrow[r] \arrow[d, dashed, "{u[1]}"] & X[1] \arrow[d, equal] \\
X \arrow[r, "f\circ\alpha"] \arrow[d, "\alpha"] & Y[1] \arrow[r] \arrow[d, equal] & K_{f\circ\alpha}[1] \arrow[r] \arrow[d, dashed, "{v[1]}"] & X[1] \arrow[d, "{\alpha[1]}"] \\
X' \arrow[r, "f"] \arrow[d] & Y[1] \arrow[r] \arrow[d] & K_f[1] \arrow[r] \arrow[d, equal] & X'[1] \arrow[d] \\
K_\alpha[1] \arrow[r, dashed] & K_{f\circ\alpha}[1] \arrow[r, dashed] & K_f[1] \arrow[r, dashed] & K_\alpha[2]
\end{tikzcd}$$

and in particular, after shifting, we have that the map $v\colon K_{f\circ\alpha} \to K_f$ sits in the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & K_{f\circ\alpha} \arrow[r] \arrow[d, "v"] & X \arrow[d, "\alpha"] \\
Y \arrow[r] & K_f \arrow[r] & X'
\end{tikzcd}$$

from which we obtain the desired map $K_{f\circ\alpha} \to K_f \times_{X'} X$. That the bottom left square in (5) commutes now follows from comparing components (i.e. using the uniqueness in the universal property of the fiber product). This proves that we have a natural transformation $\mathrm{Hom}(-, Y[1]) \to \mathrm{Ext}(-, Y)$.

We use a similar argument to obtain the naturality of $\eta_{(-)} \colon \mathrm{Hom}(X, (-)[1]) \to \mathrm{Ext}(X, -)$ for a fixed X. In particular, consider a map $\beta \colon Y \to Y'$. We must show that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(X, Y[1]) \arrow[r, "\eta_Y"] \arrow[d, "(\beta[1])_*"] & \mathrm{Ext}(X, Y) \arrow[d, "(\beta[1])_*"] \\
\mathrm{Hom}(X, Y'[1]) \arrow[r, "\eta_{Y'}"] & \mathrm{Ext}(X, Y')
\end{tikzcd}$$

commutes. Like before, the extensions $\eta_Y(f)$, $(\beta[1])_*\eta_Y(f)$, and $\eta_{Y'}(\beta_* f)$ sit as the first, second, and third row in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\beta"'] & K_f \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r] \arrow[d, equal] & K_f \sqcup_Y Y' \arrow[r] \arrow[d, dotted, "?"'] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r] & K_{\beta[1]\circ f} \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

We now apply (TR4) to f and $\beta[1]$, that is the triangles

$$\left\{ \begin{array}{ccccccc}
X & \xrightarrow{f} & Y[1] & \longrightarrow & K_f[1] & \longrightarrow & X[1] \\
X & \xrightarrow{\beta[1]\circ f} & Y'[1] & \longrightarrow & K_{\beta[1]\circ f}[1] & \longrightarrow & X[1] \\
Y[1] & \xrightarrow{\beta[1]} & Y'[1] & \longrightarrow & K_{\beta[1]}[1] & \longrightarrow & Y[2]
\end{array} \right.$$

to obtain the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y[1] \arrow[r] \arrow[d, "\beta[1]"] & K_f[1] \arrow[r] \arrow[d, dashed, "u[1]"] & X[1] \arrow[d, equal] \\
X \arrow[r, "\beta[1]\circ f"] \arrow[d, "f"] & Y'[1] \arrow[r] \arrow[d, equal] & K_{\beta[1]\circ f}[1] \arrow[r] \arrow[d, dashed, "v[1]"] & X[1] \arrow[d, "f[1]"] \\
Y[1] \arrow[r, "\beta[1]"] \arrow[d] & Y'[1] \arrow[r] \arrow[d] & K_{\beta[1]}[1] \arrow[r] \arrow[d, equal] & Y[2] \arrow[d] \\
K_f[1] \arrow[r, dashed] & K_{\beta[1]\circ f}[1] \arrow[r, dashed] & K_{\beta[1]}[1] \arrow[r, dashed] & K_f[2]
\end{tikzcd}$$

where the two top rows allow us to provide the desired map $K_f \to K_{\beta[1]\circ f}$. This shows that $\eta_{Y'}((\beta[1])_* f) = (\beta[1])_*\eta_Y(f)$. Thus we see that we have a natural transformation $\mathrm{Hom}(X, (-)[1]) \to \mathrm{Ext}(X, -)$. This completes the proof that the map $\mathrm{Hom}(X, Y[1]) \to \mathrm{Ext}(X, Y)$ is natural.

For any X, Y, denote the map $\operatorname{Hom}(X, Y[1]) \to \operatorname{Ext}(X, Y)$ by ρ^X_Y, or just ρ when X and Y are obvious. We will now show that this is a group homomorphism. Let $f, g \in \operatorname{Hom}(X, Y[1])$. The addition in $\mathcal{D}^\heartsuit$ allows us to write $f + g$ as the composition

$$X \xrightarrow{\Delta_X} X \oplus X \xrightarrow{f \oplus g} Y[1] \oplus Y[1] \xrightarrow{\nabla_Y} Y[1]$$

or, more concisely, $f + g = \nabla_Y \circ (f \oplus g) \circ \Delta_X = (\nabla_Y)_*(\Delta_X)^*(f \oplus g)$. We now note that the addition for $\mathcal{E}, \mathcal{E}' \in \operatorname{Ext}(X, Y)$ is defined by $\mathcal{E} + \mathcal{E}' = (\nabla_Y)_*(\Delta_X)^*(\mathcal{E} \oplus \mathcal{E}')$. Therefore, supposing we know that $\rho(f \oplus g) = \rho(f) \oplus \rho(g)$, we can calculate

$$\begin{aligned} \rho(f) + \rho(g) &= (\nabla_Y)_*(\Delta_X)^*(\rho(f) \oplus \rho(g)) \\ &= (\nabla_Y)_*(\Delta_X)^*\rho(f \oplus g) \\ &= \rho((\nabla_Y)_*(\Delta_X)^*(f \oplus g)) = \rho(f + g) \end{aligned}$$

using naturality. Thus it suffices to show that $\rho(f \oplus g) = \rho(f) \oplus \rho(g)$. However, this follows by applying Proposition 3.22, (TR3), and Proposition 3.17 to see that the dashed arrow in the commutative diagram

$$\begin{tikzcd}
Y \oplus Y \arrow[r] \arrow[d, equal] & K_{f \oplus g} \arrow[r] \arrow[d, dashed] & X \oplus X \arrow[r, "f \oplus g"] \arrow[d, equal] & Y[1] \oplus Y[1] \arrow[d, equal] \\
Y \oplus Y \arrow[r] & K_f \oplus K_g \arrow[r] & X \oplus X \arrow[r, "f \oplus g"] & Y[1] \oplus Y[1]
\end{tikzcd}$$

is an isomorphism. Therefore ρ^X_Y is a group homomorphism for all X, Y.

It remains to show that ρ^X_Y is bijective. Let $f \in \operatorname{Hom}(X, Y[1])$ and suppose that $\rho(f) = 0$, i.e. that the extension

$$0 \longrightarrow Y \longrightarrow K_f \longrightarrow X \longrightarrow 0$$

is split. We then have an isomorphism $\phi \colon K_f \xrightarrow{\sim} Y \oplus X$ which sits in the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & K_f \arrow[r] \arrow[d, "\phi"] & X \arrow[r, "f"] \arrow[d, equal] & Y[1] \arrow[d, dashed, "\psi"] \\
Y \arrow[r] & Y \oplus X \arrow[r] & X \arrow[r, "0"] & Y[1]
\end{tikzcd}$$

where the map ψ exists by (TR3) and is an isomorphism by Proposition 3.17. We then have that $\psi \circ f = 0$, so $f = 0$ since ψ is an isomorphism (hence a monomorphism). Therefore, ρ is injective.

To see that ρ is surjective, suppose we have an extension $\mathcal{E} \in \operatorname{Ext}(X, Y)$ given by

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0.$$

Then Corollary 5.27 tells us that there is a unique map $h \colon X \to Y[1]$ such that

$$Y \longrightarrow E \longrightarrow X \xrightarrow{h} Y[1]$$

is a distinguished triangle. We now see that $\rho(h) = \mathcal{E}$, so that ρ is surjective. ■

Corollary 5.31. *Let $X, Y \in \mathcal{D}^{\heartsuit}$. Then every extension*

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

splits if and only if $\mathrm{Hom}(X, Y[1]) = 0$.

Proof. Every extension splits precisely when $\mathrm{Hom}(X, Y[1]) = \mathrm{Ext}(X, Y) = 0$. ■

Using Theorem 5.30, we can obtain a version of the long exact sequence for Ext. In particular, one can make the definition

Definition 5.32. Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. For all pairs of objects $X, Y \in \mathcal{D}^{\heartsuit}$, we set

$$\mathrm{Ext}^i(X, Y) := \mathrm{Hom}(X, Y[i]).$$

Trivially, $\mathrm{Ext}^0 = \mathrm{Hom}$, and Theorem 5.30 says exactly that $\mathrm{Ext}^1 = \mathrm{Ext}$, so Corollary 5.27 together with Proposition 3.15 immediately gives that from a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

we obtain a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(E, X) \arrow[r] & \mathrm{Hom}(E, Y) \arrow[r] & \mathrm{Hom}(E, Z) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \mathrm{Ext}(E, X) \arrow[r] & \mathrm{Ext}(E, Y) \arrow[r] & \mathrm{Ext}(E, Z) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \mathrm{Ext}^2(E, X) \arrow[r] & \mathrm{Ext}^2(E, Y) \arrow[r] & \mathrm{Ext}^2(E, Z) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \cdots & &
\end{tikzcd}$$

which, at least for the first two rows, agrees with the one from standard homological algebra. Yoneda, in [Yon60], defines not only a version of Ext^1 independent of having enough injectives, but also a version of Ext^i in general which does not depend on injectives. It is possible to show that these agree with "ordinary" Ext-groups (this is done for modules in, for example, [Yon54], and more generally for *exact categories* in [FS10, Thm. 6.42, Thm. 6.43, Remark 6.44]) when the Abelian category is nice enough to define them. However, it is *not true* that the Ext-groups defined above (using Hom in $\mathcal{D}^{\heartsuit}$) agrees with all the Yoneda Ext-groups. What *is* true is that $\mathrm{Ext}^i(X, Y)$ is given by $\mathrm{Hom}_{\mathbf{D}(\mathcal{D}^{\heartsuit})}(X, Y[i])$. In other words, the failure will arise from the difference between $\mathcal{D}$ itself and the derived category $\mathbf{D}(\mathcal{D}^{\heartsuit})$. There are various ways of studying how these two categories relate to each other, and one example for the interested reader is through realisation functors; see [PV17] for one possible survey. Here, essentially one further endows the objects of a triangulated category with "filtrations" and uses this additional information to do the comparison.

In any case, we see that in a triangulated category $\mathcal{D}$ with a t-structure, the Ext-groups $\mathrm{Ext}^i(X, Y)$ in the heart $\mathcal{D}^{\heartsuit}$ are genuinely given by $\mathrm{Hom}(X, Y[i])$ at least for $i = 0, 1$, so this structure is determined in a sense by the ambient triangulated category.

5.5 COHOMOLOGY IS COHOMOLOGICAL

We have seen that any short exact sequence in the heart gives rise to a unique distinguished triangle in the ambient category. Similarly, any distinguished triangle contained in the heart gives rise to a short exact sequence. However, if we weaken this and allow an arbitrary distinguished triangle in the ambient triangulated category, it is not true: not every distinguished triangle in a triangulated category with a t-structure gives rise to a short exact sequence in the heart. However, we have a partial converse which illustrates that the cohomology functors H^n defined earlier behave precisely how we wish them to. In particular, by applying cohomology, we obtain a long exact sequence from any distinguished triangle.

Theorem 5.33. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then the cohomology functors $\mathrm{H}^n \colon \mathcal{D} \to \mathcal{D}^{\heartsuit}$ are cohomological.*

Proof. Due to how H^n is defined, it suffices to show that H^0 is cohomological. The proof proceeds in three steps. First, we will show the result when in a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

we know that $X, Y, Z \in \mathcal{D}^{\leq 0}$ (or, dually, $X, Y, Z \in \mathcal{D}^{\geq 0}$). The second step is to make two out of X, Y and Z arbitrary, and the final step is to have a completely arbitrary distinguished triangle.

Suppose we have a distinguished triangle as above, and suppose $X, Y, Z \in \mathcal{D}^{\leq 0}$. We need to show that

$$\mathrm{H}^0(X) \longrightarrow \mathrm{H}^0(Y) \longrightarrow \mathrm{H}^0(Z) \longrightarrow 0$$

is exact in $\mathcal{D}^{\heartsuit}$. Let $W \in \mathcal{D}^{\heartsuit}$. Applying $\mathrm{Hom}(-, W)$, we have canonical isomorphisms

$$\mathrm{Hom}(\mathrm{H}^0(Z), W) = \mathrm{Hom}(\tau^{\geq 0}\tau^{\leq 0}Z, W) \cong \mathrm{Hom}(\tau^{\leq 0}Z, W) \cong \mathrm{Hom}(Z, W)$$

and similarly for X and Y, while $\mathrm{Hom}(X[1], W) = 0$ (since $X[1] \in \mathcal{D}^{\leq -1}$ and $W \in \mathcal{D}^{\geq 0}$). From this, we deduce that we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(Z, W) \longrightarrow \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(X, W)$$

which from the prior computation gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathrm{H}^0(Z), W) \longrightarrow \mathrm{Hom}(\mathrm{H}^0(Y), W) \longrightarrow \mathrm{Hom}(\mathrm{H}^0(X), W).$$

A dual proof shows that when $X, Y, Z \in \mathcal{D}^{\geq 0}$, we have the exact sequence

$$0 \longrightarrow \mathrm{H}^0(X) \longrightarrow \mathrm{H}^0(Y) \longrightarrow \mathrm{H}^0(Z).$$

Now suppose that X, Y are arbitrary, while $Z \in \mathcal{D}^{\geq 0}$. We again show that the above diagram is exact. Let $W \in \mathcal{D}^{\leq -1}$. Then

$$\mathrm{Hom}(W, Z) = 0 \quad \text{and} \quad \mathrm{Hom}(W, Z[-1]) = 0$$

so the long exact sequence we get after applying Hom gives a natural isomorphism $\mathrm{Hom}(W, X) \xrightarrow{\sim} \mathrm{Hom}(W, Y)$. We then have isomorphisms

$$\mathrm{Hom}(W, \tau^{\leq -1}X) \cong \mathrm{Hom}(W, X) \xrightarrow{\sim} \mathrm{Hom}(W, Y) \cong \mathrm{Hom}(W, \tau^{\leq -1}Y)$$

so we have an isomorphism $\tau^{\leq -1}X \xrightarrow{\sim} \tau^{\leq -1}Y$. In order to reduce to the case in the first step, we want to obtain a distinguished triangle of the form

$$\tau^{\geq 0}X \longrightarrow \tau^{\geq 0}Y \longrightarrow Z \longrightarrow (\tau^{\geq 0}X)[1].$$

To do this, we use (TR4). In particular, we use that distinguished triangles are closed under isomorphism, so we have the distinguished triangles

$$\left\{\begin{array}{ccccccc} \tau^{\leq -1}X & \longrightarrow & X & \longrightarrow & \tau^{\geq 0}X & \longrightarrow & (\tau^{\leq -1}X)[1] \\ \tau^{\leq -1}X & \longrightarrow & Y & \longrightarrow & \tau^{\geq 0}Y & \longrightarrow & (\tau^{\leq -1}X)[1] \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}\right.$$

and get the diagram

$$\begin{tikzcd}
\tau^{\leq -1}X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \tau^{\geq 0}X \arrow[r] \arrow[d, dashed] & (\tau^{\leq -1}X)[1] \arrow[d, equal] \\
\tau^{\leq -1}X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & \tau^{\geq 0}Y \arrow[r] \arrow[d, dashed] & (\tau^{\leq -1}X)[1] \arrow[d] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[dl] \\
 & & (\tau^{\geq 0}X)[1] &
\end{tikzcd}$$

where the dashed arrows give the desired distinguished triangle. Since $\tau^{\geq 0}X, \tau^{\geq 0}Y \in \mathcal{D}^{\geq 0}$, we have natural isomorphisms $\mathrm{H}^0(X) \cong \mathrm{H}^0(\tau^{\geq 0}X)$ and $\mathrm{H}^0(Y) \cong \mathrm{H}^0(\tau^{\geq 0}Y)$. Therefore, applying H^0 gives the desired exact sequence

$$0 \longrightarrow \mathrm{H}^0(X) \longrightarrow \mathrm{H}^0(Y) \longrightarrow \mathrm{H}^0(Z).$$

An essentially identical computation shows that if Y and Z are arbitrary while $X \in \mathcal{D}^{\leq 0}$, we get the exact sequence

$$\mathrm{H}^0(X) \longrightarrow \mathrm{H}^0(Y) \longrightarrow \mathrm{H}^0(Z) \longrightarrow 0.$$

Finally, consider the case when X, Y, and Z are totally arbitrary. We apply (TR4) to the composition

$$\tau^{\leq 0}X \longrightarrow X \longrightarrow Y,$$

that is, to the distinguished triangles

$$\left\{\begin{array}{ccccccc} \tau^{\leq 0}X & \longrightarrow & X & \longrightarrow & \tau^{\geq 1}X & \longrightarrow & (\tau^{\leq 0}X)[1] \\ \tau^{\leq 0}X & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & (\tau^{\leq 0}X)[1] \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}\right.$$

where C is a cone of the given composition. Applying (TR4) then gives the diagram

$$\begin{tikzcd}
 & & & {} & & \\
 & & \tau^{\geq 1}X \arrow[ur] \arrow[dr, dashed] & & {} & \\
 & X \arrow[ur] \arrow[dr] & & C \arrow[ur] \arrow[ddr, dashed] & & \\
 & & Y \arrow[ur] \arrow[drr] & & & \\
\tau^{\leq 0}X \arrow[uur] \arrow[urr] & & & & Z \arrow[dr] \arrow[ddr, dashed] & \\
 & & & & & {} \\
 & & & & & {}
\end{tikzcd}$$

which contains two distinguished triangles of interest, namely

$$\begin{cases} \tau^{\leq 0}X & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & (\tau^{\leq 0}X)[1], \\ \tau^{\geq 1}X & \longrightarrow & C & \longrightarrow & Z & \longrightarrow & (\tau^{\geq 1}X)[1]. \end{cases}$$

Using the previously established results, the first triangle above gives that we have the exact sequence

$$\mathrm{H}^0(X) \longrightarrow \mathrm{H}^0(Y) \longrightarrow \mathrm{H}^0(C) \longrightarrow 0$$

and, after shifting the second triangle by one, we have the exact sequence

$$0 \longrightarrow \mathrm{H}^0(C) \longrightarrow \mathrm{H}^0(Z) \longrightarrow \mathrm{H}^0((\tau^{\geq 1}X)[1])$$

which, together with the commutativity of the big diagram above, lets us conclude that we have the exact sequence

$$\mathrm{H}^0(X) \longrightarrow \mathrm{H}^0(Y) \longrightarrow \mathrm{H}^0(Z).$$

More precisely, the first exact sequence gives that the map $\mathrm{H}^0(Y) \to \mathrm{H}^0(C)$ is an epimorphism, and the second one gives that the map $\mathrm{H}^0(C) \to \mathrm{H}^0(Z)$ is a monomorphism. Finally, the diagram above gives that the composition

$$\mathrm{H}^0(Y) \twoheadrightarrow \mathrm{H}^0(C) \hookrightarrow \mathrm{H}^0(Z)$$

is exactly the map $\mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$. Thus we conclude the exactness of the given sequence since the kernels of

$$\mathrm{H}^0(Y) \longrightarrow \mathrm{H}^0(C) \quad \text{and} \quad \mathrm{H}^0(Y) \longrightarrow \mathrm{H}^0(Z)$$

are the same. ■

Note that for any $X \in \mathcal{D}^{\heartsuit}$, we have a natural isomorphism $\mathrm{H}^0(X) \cong X$. Therefore, the above theorem specializes to the fact that any distinguished triangle in the heart gives a short exact sequence in the heart. Theorem 5.33 also allows us to deduce a version of the long exact sequence in cohomology. In particular, we have

Corollary 5.34. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be triangulated categories with t-structures $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$, and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a triangulated functor. Then $\mathrm{H}^0 \circ F : \mathcal{D}_1 \to \mathcal{D}_2^{\heartsuit}$ is cohomological, i.e. for any distinguished triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{-1}(F(Z)) \longrightarrow \mathrm{H}^0(F(X)) \longrightarrow \mathrm{H}^0(F(Y)) \longrightarrow \mathrm{H}^0(F(Z)) \longrightarrow \mathrm{H}^1(F(X)) \longrightarrow \cdots$$

Proof. Since F is triangulated, the triangle

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(X)[1]$$

is distinguished. Applying H^0 and using Theorem 5.33 gives the result since $\mathrm{H}^n = \mathrm{H}^0 \circ [n]$ and since we have the natural isomorphism $F \circ [1] \cong [1] \circ F$. ■

5.6 t-EXACTNESS & GLUING t-STRUCTURES

We have a suitable notion of functors of triangulated categories, namely that of a triangulated functor (Definition 3.9). However, such a functor may not preserve a given t-structure, so it is necessary to introduce the following terminology:

Definition 5.35. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be triangulated categories with t-structures $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$. A triangulated functor $F \colon \mathcal{D}_1 \to \mathcal{D}_2$ is *left (resp. right) t-exact* if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$ (resp. $F(\mathcal{D}_1^{\leq 0}) \subseteq \mathcal{D}_2^{\leq 0}$). A triangulated functor is *t-exact* if it both left and right t-exact.

For any triangulated functor $\mathcal{D}_1 \to \mathcal{D}_2$, we can build a functor $\mathcal{D}_1^{\heartsuit} \to \mathcal{D}_2^{\heartsuit}$ using cohomology as follows:

Definition 5.36. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be triangulated categories with t-structures $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$, and let $F \colon \mathcal{D}_1 \to \mathcal{D}_2$ be a triangulated functor. Then we define

$$^{p}F \colon \mathcal{D}_1^{\heartsuit} \to \mathcal{D}_2^{\heartsuit}, \quad {}^{p}F := \mathrm{H}^0 \circ F \circ \iota$$

where $\iota \colon \mathcal{D}_1^{\heartsuit} \to \mathcal{D}_1$ is the inclusion.

Remark 5.37. Note that Corollary 5.34, together with Corollary 5.27, states that for any exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

in $\mathcal{D}_1^{\heartsuit}$, applying ${}^{p}F$ (specifically, to the induced distinguished trianglei in $\mathcal{D}_1$) gives us a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{-1}(F(Z)) \longrightarrow {}^{p}F(X) \longrightarrow {}^{p}F(Y) \longrightarrow {}^{p}F(Z) \longrightarrow \mathrm{H}^1(F(X)) \longrightarrow \cdots$$

in $\mathcal{D}_2^{\heartsuit}$.

In general, one does not know too much about ${}^{p}F$, but when in addition F has some t-exactness condition it is possible to deduce some more properties.

Lemma 5.38. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be as above, and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a left (resp. right) t-exact functor. Then for all $n \in \mathbb{Z}$, we have that $F(\mathcal{D}_1^{\geq n}) \subseteq \mathcal{D}_2^{\geq n}$ (resp. $F(\mathcal{D}_1^{\leq n}) \subseteq \mathcal{D}_2^{\leq n}$).*

Proof. We prove this for the case when F is right t-exact, since the left t-exact case is dual. We have that $\mathcal{D}_1^{\leq n} = \mathcal{D}_1^{\leq 0}[-n]$. Since F is triangulated and right t-exact, we have that

$$F(\mathcal{D}_1^{\leq n}) = F(\mathcal{D}_1^{\leq 0}[-n]) = F(\mathcal{D}_1^{\leq 0})[-n] \subseteq \mathcal{D}_2^{\leq 0}[-n] = \mathcal{D}_2^{\leq n}$$

as desired. ■

Proposition 5.39. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be as above, and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a left (resp. right) t-exact functor. Then*

(i) for all $X \in \mathcal{D}_1^{\geq 0}$ (resp. $\mathcal{D}_1^{\leq 0}$), we have $\mathrm{H}^0(F(X)) \cong {}^{p}F(\mathrm{H}^0(X))$, and

(ii) ${}^{p}F$ is a left (resp. right) exact functor.

Proof. We consider the case when F is right t-exact, since the other case follows by a dual argument.

We begin with (i). Take $X \in \mathcal{D}_1^{\leq 0}$, and note that since $\tau^{\leq 0}X \cong X$, we have that $\mathrm{H}^0(X) \cong \tau^{\geq 0}X$. Therefore, we have a distinguished triangle

$$\tau^{\leq -1}X \longrightarrow X \longrightarrow \mathrm{H}^0(X) \longrightarrow (\tau^{\leq -1}X)[1].$$

Since F is right t-exact we have that $F(\tau^{\leq -1}X) \in \mathcal{D}_2^{\leq -1}$ so that Proposition 5.26 lets us compute $\mathrm{H}^0(F(\tau^{\leq -1}X)) \cong \mathrm{H}^1(F(\tau^{\leq -1}X)) = 0$, which means that Corollary 5.34 gives us an exact sequence

$$0 \longrightarrow \mathrm{H}^0(F(X)) \longrightarrow \mathrm{H}^0(F(\mathrm{H}^0(X))) \longrightarrow 0$$

so that $\mathrm{H}^0(F(X)) \cong {}^pF(\mathrm{H}^0(X))$.

To prove (ii), consider an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{D}_1^{\heartsuit}$. As in Remark 5.37, Corollary 5.34 then produces the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{-1}(F(Z)) \longrightarrow {}^pF(X) \longrightarrow {}^pF(Y) \longrightarrow {}^pF(Z) \longrightarrow \mathrm{H}^1(F(X)) \longrightarrow \cdots$$

and since F is right t-exact and $X, Y, Z \in \mathcal{D}_1^{\heartsuit} = \mathcal{D}_1^{\leq 0} \cap \mathcal{D}_1^{\geq 0}$, we see that $F(X) \in \mathcal{D}_2^{\leq 0}$. Using Proposition 5.26, we get that $\mathrm{H}^1(F(X)) = 0$, so we can then reduce the above long exact sequence to

$${}^pF(X) \longrightarrow {}^pF(Y) \longrightarrow {}^pF(Z) \longrightarrow 0.$$

This proves the result. ■

Corollary 5.40. *Let $\mathcal{D}_1$, $\mathcal{D}_2$, and $F \colon \mathcal{D}_1 \to \mathcal{D}_2$ be as above. If F is t-exact, then $F(\mathcal{D}_1^{\heartsuit}) \subseteq \mathcal{D}_2^{\heartsuit}$, ${}^pF$ is exact, we have a natural isomorphism $F|_{\mathcal{D}_1^{\heartsuit}} \cong {}^pF$, and*

$$F(\mathrm{H}^n(X)) = \mathrm{H}^n(F(X))$$

for all $X \in \mathcal{D}_1$.

Given that we now have a notion of compatibility for functors between triangulated categories with t-structures, we can begin to discuss the topic of gluing t-structures. Here, we essentially follow the outline from [GM03, p. 286, Ex. IV.4.2].

Definition 5.41. A Verdier quotient sequence

$$\mathcal{C} \xhookrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}$$

where $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ have t-structures is *compatible* if P and Q are t-exact.

We then make the following observation:

Proposition 5.42. *Consider a compatible Verdier quotient sequence*

$$\mathcal{C} \xhookrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}.$$

Then

$$P(\mathcal{C}^{\leq 0}) = \mathcal{D}^{\leq 0} \cap P(\mathcal{C}), \qquad \mathcal{E}^{\leq 0} = Q(\mathcal{D}^{\leq 0}),$$
$$P(\mathcal{C}^{\geq 0}) = \mathcal{D}^{\geq 0} \cap P(\mathcal{C}), \qquad \mathcal{E}^{\geq 0} = Q(\mathcal{D}^{\geq 0}).$$

Proof. That $P(\mathcal{C}^{\leq 0}) \subseteq \mathcal{D}^{\leq 0} \cap P(\mathcal{C})$ is obvious since P is t-exact. Conversely, let $X \in \mathcal{D}^{\leq 0} \cap P(\mathcal{C})$, and say $X \cong P(X_0)$. We observe that $P(\tau^{\leq 0}X_0) \in \mathcal{D}^{\leq 0}$ and $P(\tau^{\geq 1}X_0) \in \mathcal{D}^{\geq 1}$ by t-exactness, and that we have the distinguished triangle

$$\tau^{\leq 0}X_0 \longrightarrow X_0 \longrightarrow \tau^{\geq 1}X_0 \longrightarrow (\tau^{\leq 0}X_0)[1]$$

in $\mathcal{C}$. Applying P to the this yields a distinguished triangle where the left entry is in $\mathcal{D}^{\geq 0}$, the right entry is in $\mathcal{D}^{\geq 1}$, and the middle is $P(X_0) \cong X$. However, we already have one such distinguished triangle, namely

$$\tau^{\leq 0}X \xrightarrow{\sim} X \longrightarrow 0 \longrightarrow (\tau^{\leq 0}X)[1]$$

so by uniqueness of truncation, we see that $P(\tau^{\geq 1}X_0) = 0$ which tells us that $X_0 \in \mathcal{C}^{\leq 0}$. The proof that $P(\mathcal{C}^{\geq 0}) = \mathcal{D}^{\geq 0} \cap P(\mathcal{C})$ is identical.

For the other half of the proposition, it is similarly clear that $Q(\mathcal{D}^{\leq 0}) \subseteq \mathcal{E}^{\leq 0}$ since Q is t-exact. Conversely, let $X_1 \in \mathcal{E}^{\leq 0}$. Since Q is essentially surjective, we can find some $X \in \mathcal{D}$ such that $Q(X) \cong X_1$. We then have the distinguished triangle

$$\tau^{\leq 0}X \longrightarrow X \longrightarrow \tau^{\geq 1}X \longrightarrow (\tau^{\leq 0}X)[1]$$

where we note that $Q(\tau^{\leq 0}X) \in \mathcal{E}^{\leq 0}$ and $Q(\tau^{\geq 1}X) \in \mathcal{E}^{\geq 1}$ by t-exactness. Thus, applying Q to this distinguished triangle yields a distinguished triangle where the left entry is in $\mathcal{E}^{\leq 0}$, the right entry is in $\mathcal{E}^{\geq 1}$, and the middle is $Q(X) \cong X_1$. Like before, we already know of one such distinguished triangle, namely

$$\tau^{\leq 0}X_1 \xrightarrow{\sim} X_1 \longrightarrow 0 \longrightarrow (\tau^{\leq 0}X_1)[1]$$

which by uniqueness of truncation implies that $X_1 \cong \tau^{\leq 0}X_1 \cong Q(X\tau^{\leq 0}X)$. Therefore, $X_1 \in Q(\mathcal{D}^{\leq 0})$, so $\mathcal{E}^{\leq 0} = Q(\mathcal{D}^{\leq 0})$. The proof that $\mathcal{E}^{\geq 0} = Q(\mathcal{D}^{\geq 0})$ is identical. ■

That is, in a compatible Verdier quotient sequence

$$\mathcal{C} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{E}$$

the t-structures on $\mathcal{C}$ and $\mathcal{E}$ are uniquely determined by the t-structure on $\mathcal{D}$. We also have the following:

Proposition 5.43. *Consider a compatible Verdier quotient sequence*

$$\mathcal{C} \xhookrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}.$$

Then

$$\mathcal{D}^{\leq 0} = \{X \in {}^{\perp}P(\mathcal{C}^{\geq 1}) \mid Q(X) \in \mathcal{E}^{\leq 0}\}, \text{ and}$$
$$\mathcal{D}^{\geq 0} = \{X \in P(\mathcal{C}^{\leq -1})^{\perp} \mid Q(X) \in \mathcal{E}^{\geq 0}\}.$$

To prove this, we first need a lemma.

Lemma 5.44. *Consider a compatible Verdier quotient sequence*

$$\mathcal{C} \xhookrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}.$$

Then for any $Y \in \mathcal{D}$, we have $Q(Y) \in \mathcal{E}^{\leq 0}$ (resp. $Q(Y) \in \mathcal{E}^{\geq 0}$) if and only if $\tau^{\geq 1}Y \in P(\mathcal{C}^{\geq 1})$ (resp. $\tau^{\leq -1}Y \in P(\mathcal{C}^{\leq -1})$).

Proof. The sufficiency of the latter condition is clear. For the other direction, we have the distinguished triangle

$$\tau^{\le 0}Y \longrightarrow Y \longrightarrow \tau^{\ge 1}Y \longrightarrow (\tau^{\le 0}Y)[1]$$

which is mapped to the distinguished triangle

$$\tau^{\le 0}Q(Y) \longrightarrow Q(Y) \longrightarrow 0 \longrightarrow (\tau^{\le 0}Q(Y))[1]$$

in $\mathcal{E}$ by Q. Therefore, $Q(\tau^{\ge 1}Y) = 0$, so

$$\tau^{\ge 1}Y \in \mathcal{D}^{\ge 1} \cap \ker Q = \mathcal{D}^{\ge 1} \cap P(\mathcal{C}) = P(\mathcal{C}^{\ge 1})$$

which completes the proof after recognizing that the other assertion is dual. ■

With the above lemma in place, we can move on to a proof of the proposition.

Proof of Proposition 5.43. Let $X \in \mathcal{D}^{\le 0}$. Is is then clear that $\mathrm{Hom}_{\mathcal{D}}(X, P(Y_0)) = 0$ for all $Y_0 \in \mathcal{C}^{\ge 1}$ since $P(\mathcal{C}^{\ge 1}) \subseteq \mathcal{D}^{\ge 1}$. Similarly, $Q(X) \in \mathcal{E}^{\le 0}$ since $Q(\mathcal{D}^{\le 0}) = \mathcal{E}^{\le 0}$. This shows one of the desired inclusions. Conversely, suppose $X \in \mathcal{D}$ satisfies $X \in {}^{\perp}P(\mathcal{C}^{\ge 1})$ and $Q(X) \in \mathcal{E}^{\le 0}$. By Lemma 5.44, the latter condition implies that $\tau^{\ge 1}X \in P(\mathcal{C}^{\ge 1})$. Therefore, we observe that $\mathrm{Hom}_{\mathcal{D}}(X, \tau^{\ge 1}X) = 0$, which by Lemma 5.17 implies that $X \in \mathcal{D}^{\le 0}$. This shows that

$$\mathcal{D}^{\le 0} = \{X \in {}^{\perp}P(\mathcal{C}^{\ge 1}) \mid Q(X) \in \mathcal{E}^{\le 0}\}$$

as desired. The other half of the proposition is dual. ■

The point of the above two propositions is to say that if we are given a compatible Verdier quotient sequence

$$\mathcal{C} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{E}$$

then we can totally recover the t-structures on $\mathcal{C}$ and $\mathcal{E}$ from the one on $\mathcal{D}$, and similarly, we can totally recover the t-structure on $\mathcal{D}$ from the ones on $\mathcal{C}$ and $\mathcal{E}$. A notable issue this has, however, is that it assumes a priori that there are t-structures on all the involved triangulated categories, and furthermore that we have this compatibility condition of the functors being t-exact.

A way of interpreting the above statements is to say that in such a compatible Verdier quotient sequence, the t-structure on $\mathcal{D}$ is necessarily "glued together" from the ones on $\mathcal{C}$ and $\mathcal{E}$. The limitation is that if we do not know from the start that there exists a t-structure on $\mathcal{D}$, we cannot actually conclude anything at all in general. This is to be expected: if we drop the t-structure on $\mathcal{D}$, then we also drop the compatibility condition, and therefore we have no guarantee that the t-structures on $\mathcal{C}$ and $\mathcal{E}$ interact well with each other.

A way to produce a situation where we *can* always "glue" t-structures is to ensure that the Verdier quotient sequence is actually doing some gluing, i.e. that it is a recollement. See Section 3.5 for details about those. We then have

Theorem 5.45. *Suppose there is a recollement*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P" description] & \mathcal{D} \arrow[l, two heads, bend right, "L_P"'] \arrow[l, two heads, bend left, "R_P"] \arrow[r, two heads, "Q" description] & \mathcal{E}. \arrow[l, hook, bend right, "L_Q"'] \arrow[l, hook, bend left, "R_Q"]
\end{tikzcd}$$

and suppose that $\mathcal{C}$ and $\mathcal{E}$ have t-structures on them. Then there exists a t-structure on $\mathcal{D}$ such that the above Verdier quotient sequence is compatible. Furthermore, the t-structure is given by

$$\mathcal{D}^{\le 0} = \{X \in {}^{\perp}P(\mathcal{C}^{\ge 1}) \ \mid Q(X) \in \mathcal{E}^{\le 0}\}, \ \textit{and}$$
$$\mathcal{D}^{\ge 0} = \{X \in P(\mathcal{C}^{\le -1})^{\perp} \mid Q(X) \in \mathcal{E}^{\ge 0}\}.$$

Before we prove this, we collect some lemmas regarding the situation in the theorem.

Lemma 5.46. *With the hypotheses of Theorem 5.45, we have*

$$L_P(\mathcal{D}^{\leq 0}) = \mathcal{C}^{\leq 0}, \quad R_P(\mathcal{D}^{\geq 0}) = \mathcal{C}^{\geq 0},$$

and

$$P(\mathcal{C}^{\leq 0}) \subseteq \mathcal{D}^{\leq 0}, \quad P(\mathcal{C}^{\geq 0}) \subseteq \mathcal{D}^{\geq 0}.$$

Proof. The last part of the lemma is very easy: if $X' \in \mathcal{C}^{\leq 0}$, then $Q(P(X')) = 0 \in \mathcal{E}^{\leq 0}$ and for all $Y' \in \mathcal{C}^{\geq 1}$ we have

$$\mathrm{Hom}_{\mathcal{D}}(P(X'), P(Y')) \cong \mathrm{Hom}_{\mathcal{C}}(X', Y') = 0$$

so $P(X') \in \mathcal{D}^{\leq 0}$. The other argument is dual. With that taken care of, we do the first part.

Let $X \in \mathcal{D}^{\leq 0}$. Then, for any $Y' \in \mathcal{C}^{\geq 1}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(L_P(X), Y') \cong \mathrm{Hom}_{\mathcal{D}}(X, P(Y')) = 0$$

since $X \in\perp P(\mathcal{C}^{\geq 1})$, so $L_P(X) \in \mathcal{C}^{\leq 0}$. Therefore, $L_P(\mathcal{D}^{\leq 0}) \subseteq \mathcal{C}^{\leq 0}$. Next, if $X' \in \mathcal{C}^{\leq 0}$, then since P is fully faithful we have $L_P(P(X')) \cong X'$. Furthermore, $P(X') \in \mathcal{D}^{\leq 0}$ since $Q(P(X')) = 0 \in \mathcal{E}^{\leq 0}$ and if $Y \in P(\mathcal{C}^{\geq 1})$ (i.e. $Y \cong P(Y')$, with $Y' \in \mathcal{C}^{\geq 1}$) then

$$\mathrm{Hom}_{\mathcal{D}}(P(X'), Y) \cong \mathrm{Hom}_{\mathcal{D}}(P(X'), P(Y')) \cong \mathrm{Hom}_{\mathcal{C}}(X', Y') = 0$$

since $Y' \in \mathcal{C}^{\geq 1}$. Therefore, $\mathcal{C}^{\leq 0} \subseteq L_P(\mathcal{D}^{\leq 0})$, and so we have $\mathcal{C}^{\leq 0} = L_P(\mathcal{D}^{\leq 0})$.

An identical but dual argument shows that $R_P(\mathcal{D}^{\geq 0}) = \mathcal{C}^{\geq 0}$. ■

Lemma 5.47. *With the hypotheses of Theorem 5.45, we have*

$$L_Q(\mathcal{E}^{\leq 0}) \subseteq \mathcal{D}^{\leq 0}, \quad R_Q(\mathcal{E}^{\geq 0}) \subseteq \mathcal{D}^{\geq 0},$$

and

$$Q(\mathcal{D}^{\leq 0}) = \mathcal{E}^{\leq 0}, \quad Q(\mathcal{D}^{\geq 0}) = \mathcal{E}^{\geq 0}.$$

Proof. If $X'' \in \mathcal{E}^{\leq 0}$, then $Q(L_Q(X'')) \cong X''$ and for all $Y' \in \mathcal{C}^{\geq 1}$ we have

$$\mathrm{Hom}_{\mathcal{D}}(L_Q(X''), P(Y')) \cong \mathrm{Hom}_{\mathcal{E}}(X'', Q(P(Y'))) = 0$$

so $L_Q(X'') \in \mathcal{D}^{\leq 0}$. Therefore, $L_Q(\mathcal{E}^{\leq 0}) \subseteq \mathcal{D}^{\leq 0}$.

The second part follows by the observation that trivially $Q(\mathcal{D}^{\leq 0}) \subseteq \mathcal{E}^{\leq 0}$ by the definition of $\mathcal{D}^{\leq 0}$, and the restrictions $L_Q|_{\mathcal{E}^{\leq 0}}$ and $Q|_{\mathcal{D}^{\leq 0}}$ are adjoint to each other. Since the restriction of L_Q is fully faithful, the restriction of Q is essentially surjective. ■

The above two lemmas, and essentially all the information about the situation, can be summarized in the following picture:

$$\begin{tikzcd}
\mathcal{C}^{\leq 0} \arrow[r, shift right, hook] \arrow[d, shift left, hook] & \mathcal{D}^{\leq 0} \arrow[l, shift right, two heads] \arrow[r, shift right, two heads] \arrow[d, hook] & \mathcal{E}^{\leq 0} \arrow[l, shift right, hook] \arrow[d, shift left, hook] \\
\mathcal{C} \arrow[u, shift left, two heads, "\tau^{\leq 0}"] \arrow[r, hook, "P" description] \arrow[d, shift right, two heads, "\tau^{\geq 0}"'] & \mathcal{D} \arrow[l, bend right, two heads, "L_P"'] \arrow[l, bend left, two heads, "R_P"] \arrow[r, two heads, "Q" description] & \mathcal{E} \arrow[l, bend right, hook, "L_Q"'] \arrow[l, bend left, hook, "R_Q"] \arrow[u, shift left, two heads, "\tau^{\leq 0}"] \arrow[d, shift right, two heads, "\tau^{\geq 0}"'] \\
\mathcal{C}^{\geq 0} \arrow[u, shift right, hook] \arrow[r, shift left, hook] & \mathcal{D}^{\geq 0} \arrow[u, hook] \arrow[l, shift left, two heads] \arrow[r, shift left, two heads] & \mathcal{E}^{\geq 0} \arrow[u, shift right, hook] \arrow[l, shift left, hook]
\end{tikzcd}$$

Take note that this does not commute.

Finally, we have one more easy lemma which will be helpful as a slightly different characterization of the prospective t-structure on $\mathcal{D}$.

Lemma 5.48. *Suppose we are in the situation of Theorem 5.45. Then we have equalities*

$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} \mid L_P(X) \in \mathcal{C}^{\leq 0},\ Q(X) \in \mathcal{E}^{\leq 0}\}, \text{ and}$$
$$\mathcal{D}^{\geq 0} = \{X \in \mathcal{D} \mid R_P(X) \in \mathcal{C}^{\geq 0},\ Q(X) \in \mathcal{E}^{\geq 0}\}.$$

Proof. Let $\mathcal{D}' = \{X \in \mathcal{D} \mid L_P(X) \in \mathcal{C}^{\leq 0},\ Q(X) \in \mathcal{D}^{\leq 0}\}$. If $X \in \mathcal{D}^{\leq 0}$, then by definition $Q(X) \in \mathcal{E}^{\leq 0}$. Furthermore, we see that $L_P(X) \in \mathcal{C}^{\leq 0}$ by Lemma 5.46. Therefore, $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}'$. Conversely, let $X \in \mathcal{D}'$. Again, trivially $Q(X) \in \mathcal{E}^{\leq 0}$. Now, let $Z \in \mathcal{C}^{\geq 1}$. Then

$$\mathrm{Hom}_{\mathcal{D}}(X, P(Z)) \cong \mathrm{Hom}_{\mathcal{C}}(L_P(X), Z) = 0$$

since $L_P(X) \in \mathcal{C}^{\leq 0}$. Therefore, $X \in {}^{\perp}P(\mathcal{C}^{\geq 1})$. This proves that $\mathcal{D}' \subseteq \mathcal{D}^{\leq 0}$, so $\mathcal{D}' = \mathcal{D}^{\leq 0}$. Thus we have proved one equality. The other one follows similarly. ■

With this work out of the way, we may prove the theorem.

Proof of Theorem 5.45. We begin with (T2). Let $X \in \mathcal{D}^{\leq 0}$. Then we see that

$$L_P(X[1]) \cong L_P(X)[1] \in \mathcal{C}^{\leq -1} \subseteq \mathcal{C}^{\leq 0}$$

and

$$Q(X[1]) \cong Q(X)[1] \in \mathcal{E}^{\leq -1} \subseteq \mathcal{C}^{\leq 0}$$

so $X[1] \in \mathcal{C}^{\leq 0}$. An identical calculation shows that $Y[-1] \in \mathcal{D}^{\geq 0}$ for all $Y \in \mathcal{D}^{\geq 0}$.

Now we show (T1). Suppose $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 0}$. By Corollary 3.89, we have a distinguished triangle

$$L_Q(Q(X)) \longrightarrow X \longrightarrow P(L_P(X)) \longrightarrow L_Q(Q(X))[1].$$

Applying $\mathrm{Hom}_{\mathcal{D}}(-, Y[-1])$ to this, we obtain the exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(P(L_P(X)), Y[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L_Q(Q(X)), Y[-1]).$$

We then use that L_Q is left adjoint to Q and P is left adjoint to R_P to rewrite this as

$$\mathrm{Hom}_{\mathcal{C}}(L_P(X), R_P(Y)[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(Q(X), Q(Y)[-1]).$$

By Lemma 5.48, the first and last terms are zero. Therefore, $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$.

Finally, we show (T3). Here we follow [BBDG18, Thm. 1.4.10]. Let $X \in \mathcal{D}$. Applying Q, truncating, and then apply R_Q, we have a distinguished triangle

$$R_Q(\tau^{\leq 0}Q(X)) \longrightarrow R_Q(Q(X)) \longrightarrow R_Q(\tau^{\geq 1}Q(X)) \longrightarrow R_Q(\tau^{\leq 0}Q(X))[1].$$

We compose the maps $X \to R_Q(Q(X)) \to R_Q(\tau^{\geq 1}Q(X))$ to obtain a map $X \to R_Q(\tau^{\geq 1}Q(X))$. Taking the cocone of this map, we have a distinguished triangle

$$X' \longrightarrow X \longrightarrow R_Q(\tau^{\geq 1}Q(X)) \longrightarrow X'[1].$$

Similarly we produce a map $X' \to P(\tau^{\geq 1}L_P(X'))$ and a distinguished triangle

$$A \longrightarrow X' \longrightarrow P(\tau^{\geq 1}L_P(X')) \longrightarrow A[1].$$

We now apply (TR4) to the composition $A \to X' \to X$. This produces the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d] & P(\tau^{\geq 1}L_P(X')) \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, equal] \\
A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, dashed] & A[1] \arrow[d] \\
X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & R_Q(\tau^{\geq 1}Q(X)) \arrow[r] \arrow[d, equal] & X'[1] \arrow[d] \\
P(\tau^{\geq 1}L_P(X)) \arrow[r, dashed] & B \arrow[r, dashed] & R_Q(\tau^{\geq 1}Q(X)) \arrow[r, dashed] & P(\tau^{\geq 1}L_P(X))[1]
\end{tikzcd}$$

and we may now obtain useful information by applying the functors available to us to these distinguished triangles. In particular, applying Q to the bottom triangle gives

$$0 \longrightarrow Q(B) \xrightarrow{\sim} \tau^{\geq 1}Q(X) \longrightarrow 0.$$

Applying R_P to the same triangle and using that $\ker R_P = \operatorname{im} R_Q$ gives

$$\tau^{\geq 1}L_P(X') \xrightarrow{\sim} R_P(B) \longrightarrow 0 \longrightarrow \tau^{\geq 1}L_P(X')[1].$$

This implies, by Lemma 5.48, that $B \in \mathcal{D}^{\geq 1}$. Similarly, applying L_P to the first row of the diagram gives

$$L_P(A) \longrightarrow L_P(X') \longrightarrow \tau^{\geq 1}L_P(X') \longrightarrow L_P(A)[1]$$

so by uniqueness of truncation $L_P(A) \cong \tau^{\leq 0}L_P(X')$. Applying Q to the second row of the diagram and using that $Q(B) \cong \tau^{\geq 1}Q(X)$ gives

$$Q(A) \longrightarrow Q(X) \longrightarrow \tau^{\geq 1}Q(X) \longrightarrow Q(A)[1]$$

so, again by uniqueness of truncation, we have $Q(A) \cong \tau^{\leq 0}Q(X)$. This shows, by Lemma 5.48, that $A \in \mathcal{D}^{\leq 0}$. Thus, the distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

is the desired triangle. ■

Example 5.49. We can try applying Theorem 5.45. Consider any recollement

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "P" description] & \mathcal{D} \arrow[l, bend right, "L_P"'] \arrow[l, bend left, "R_P"] \arrow[r, two heads, "Q" description] & \mathcal{E}. \arrow[l, bend right, hook, "L_Q"'] \arrow[l, bend left, hook, "R_Q"]
\end{tikzcd}$$

Recall then from Example 5.4 that on $\mathcal{C}$ and $\mathcal{E}$ there exist various trivial t-structures. One can ask what happens when we glue them. Let $t_1(\mathcal{T}) = (\mathcal{T}, 0)$, $t_2(\mathcal{T}) = (0, \mathcal{T})$ for $\mathcal{T} = \mathcal{C}, \mathcal{D}, \mathcal{E}$. We then see that gluing $t_1(\mathcal{C})$ and $t_1(\mathcal{E})$ produces the t-structure $t_1(\mathcal{D})$ on $\mathcal{D}$, and similarly for gluing $t_2(\mathcal{C})$ and $t_2(\mathcal{E})$. Thus, these two choices do not give particularly interesting results. On the other hand, the remaining two possibilities do: if we glue $t_1(\mathcal{C})$ with $t_2(\mathcal{E})$, then we get the t-structure

$$\begin{aligned}
\mathcal{D}^{\leq 0}_{1,2} &= \{X \in \mathcal{D} \mid L_P(X) \in \mathcal{C},\ Q(X) \in 0\} \\
&= \{X \in \mathcal{D} \mid Q(X) = 0\} = P(\mathcal{C}), \\
\mathcal{D}^{\geq 0}_{1,2} &= \{X \in \mathcal{D} \mid R_P(X) \in 0,\ Q(X) \in \mathcal{E}\} \\
&= \{X \in \mathcal{D} \mid R_P(X) = 0\} = P(\mathcal{C})^{\perp}.
\end{aligned}$$

This gives another proof that ${}^{\perp}(P(\mathcal{C})^{\perp}) = P(\mathcal{C})$ when we have a recollement. In particular, $P(\mathcal{C})^{\perp}[-1] = P(\mathcal{C})^{\perp}$, and thus Corollary 5.14 says that

$$P(\mathcal{C}) = \mathcal{D}^{\leq 0}_{1,2} = {}^{\perp}(\mathcal{D}^{\geq 1}_{1,2}) = {}^{\perp}(P(\mathcal{C})^{\perp}).$$

Similarly, we may glue $t_2(\mathcal{C})$ with $t_2(\mathcal{E})$ to get the t-structure

$$\begin{aligned}\mathcal{D}^{\leq 0}_{2,1} &= \ker L_P = {}^{\perp}P(\mathcal{C}),\\ \mathcal{D}^{\geq 0}_{2,1} &= \ker Q = P(\mathcal{C}).\end{aligned}$$

Essentially the same computation as before then gives an alternative proof that $({}^{\perp}P(\mathcal{C}))^{\perp} = P(\mathcal{C})$.

Example 5.50. Suppose we have a recollement as above. In general, given a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$, one obtains t-structures $(\mathcal{T}^{\leq n}, \mathcal{T}^{\geq n})$ for and $n \in \mathbb{Z}$. We can use this fact to produce some t-structures on $\mathcal{D}$ given t-structures on $\mathcal{C}$ and $\mathcal{E}$. In particular, if the latter have t-structures, then for any two pairs of integers n_1, n_2, we can produce the t-structure

$$\begin{aligned}\mathcal{D}^{\leq 0}_{n_1,n_2} &= \{X \in \mathcal{D} \mid L_P(X) \in \mathcal{C}^{\leq n_1},\ Q(X) \in \mathcal{E}^{\leq n_2}\},\\ \mathcal{D}^{\geq 0}_{n_1,n_2} &= \{X \in \mathcal{D} \mid R_P(X) \in \mathcal{C}^{\geq n_1},\ Q(X) \in \mathcal{E}^{\geq n_2}\}.\end{aligned}$$

For example, one then has $X \in \mathcal{D}^{\leq 0}_{n_1,n_2}$ if when one projects X onto $\mathcal{C}$, it is concentrated in degree $\leq n_1$, and when one projects X onto $\mathcal{E}$, it is concentrated in degree $\leq n_2$.

5.7 NOTES ON ENHANCEMENTS

In Section 3.6, we briefly discussed the benefits of using enhancements of triangulated categories, and in particular emphasized stable ∞-categories as a good example. As observed there, there are certain differences between triangulated categories and stable ∞-categories which make the latter significantly nicer in some respects (though perhaps more cumbersome to work with in others). Since t-structures are of general interest, there is an immediate question of how one applies them to the framework of a stable ∞-category.

As it turns out, somewhat remarkably, the correct notion of t-structure on a stable ∞-category is exactly a t-structure on a triangulated category. Specifically, if we have a stable ∞-category $\mathcal{C}$, then its homotopy category $\mathrm{h}(\mathcal{C})$ is a triangulated category, and a t-structure on $\mathcal{C}$ is exactly the same as a t-structure on $\mathrm{h}(\mathcal{C})$. The reason for this is explained very well in the MathOverflow answer [Tan14]. A t-structure on $\mathcal{D} = \mathrm{h}(\mathcal{C})$ consists of two full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ such that

(a) $\mathcal{D}^{\leq 0}$ is stable under shifting to the left, and $\mathcal{D}^{\geq 0}$ is stable under shifting to the right,

(b) there is a certain distinguished triangle putting any object of $\mathcal{D}$ between one in $\mathcal{D}^{\leq 0}$ and one in $\mathcal{D}^{\geq 1}$, and

(c) there is an orthogonality condition that

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0.$$

Changing all the appropriate words, one sees that a t-structure on $\mathcal{C}$ is a pair of full subcategories $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ such that

(A) the subcategories are stable under some shifting (which is a built in operation in stable ∞-categories),

(B) there is a *fiber sequence* putting any object of $\mathcal{C}$ between one in $\mathcal{C}^{\leq 0}$ and one in $\mathcal{C}^{\geq 1}$, and

(C) there is an orthogonality condition that $\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1})$ is *contractible.*

It is then not too difficult to sketch an argument for why these two collections of data are equivalent, given sufficient knowledge of stable ∞-categories. If one actually checks the relevant definitions in [Lur17], it is fairly clear why (b) and (B) are equivalent, for example.

6 t-STRUCTURES FROM SILTING OBJECTS

In Section 5, we explored the theory of t-structures on triangulated categories. We saw a few different kinds of examples, including some simple ways to build t-structures on an ambient triangulated category obtained by gluing. However, the only non-trivial "explicit" example we saw was the standard t-structure on $\mathbf{D}(\mathcal{A})$. This section is dedicated to producing a class of particularly nice examples of t-structures on triangulated categories which have a *compact silting object*. In terms of source material, this section is very heavily based on the currently unpublished notes of my advisor, Gustavo Jasso, on dg-categories [Jas21].

We begin with a motivating example.

6.1 MOTIVATION

Fix a ring R. We can associate to this an Abelian category $\mathbf{Mod}_R$, consisting of (right) modules over R. Using the machinery of Section 4, we then produce a triangulated category $\mathbf{D}(R) := \mathbf{D}(\mathbf{Mod}_R)$, which by the discussion at the start of Section 5 has a standard t-structure on it. This t-structure has the property that we may express it directly using the vanishing of certain cohomology functors (which is not true of all t-structures). Specifically, the t-structure on $\mathbf{D}(R)$ is given by

$$\mathbf{D}(R)^{\leq 0} = \{M^\bullet \in \mathbf{D}(R) \mid \forall i > 0,\, \mathrm{H}^i(M^\bullet) = 0\},$$
$$\mathbf{D}(R)^{\geq 0} = \{M^\bullet \in \mathbf{D}(R) \mid \forall i < 0,\, \mathrm{H}^i(M^\bullet) = 0\}.$$

Being able to state the t-structure on $\mathbf{D}(R)$ in terms of its cohomology functors is already rather nice, but there is an additional phenomenon in this category which will be the main motivation for this section. It is the fact that we may express the cohomology functors using Hom-functors, and thus use Hom in order to detect whether something is in the aisle/coaisle.

In all the categories $\mathbf{C}(\mathbf{Mod}_R)$, $\mathbf{K}(\mathbf{Mod}_R)$, and $\mathbf{D}(R)$, we have a notable specific object of interest, namely R itself—or more precisely, the complex with R concentrated in degree zero. We begin with the following result:

Proposition 6.1. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{K}(\mathbf{Mod}_R)}(R, M^\bullet) \xrightarrow{\sim} \mathrm{H}^0(M^\bullet), \quad f \mapsto [f(1)] \in \mathrm{H}^0(M^\bullet)$$

for all $M^\bullet \in \mathbf{K}(\mathbf{Mod}_R)$.

Proof sketch. First, note that we have

$$\mathrm{Hom}_{\mathbf{C}(\mathbf{Mod}_R)}(R, M^\bullet) = \{f\colon R \to M^\bullet \mid d_M^0 \circ f = 0\} \cong \ker d^0$$

simply via the map $f \mapsto f(1)$. Now we notice that

$$\begin{aligned} f \sim_{\mathrm{h}} 0 &\iff \exists \eta\colon R \to M^{-1} \text{ s.t } d_M^{-1} \circ \eta = f \\ &\iff f(1) \in \operatorname{im} d_M^{-1} \end{aligned}$$

and therefore we get

$$\mathrm{Hom}_{\mathbf{K}(\mathbf{Mod}_R)}(R, M^\bullet) = \mathrm{Hom}_{\mathbf{C}(\mathbf{Mod}_R)}(R, M^\bullet)/\sim_{\mathrm{h}} \xrightarrow{\sim} \ker(d_M^0)/\operatorname{im}(d_M^{-1}) = \mathrm{H}^0(M^\bullet)$$

as desired. □

The above result applies to $\mathbf{K}(\mathbf{Mod}_R)$, but we want it for $\mathbf{D}(R)$.

Proposition 6.2. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(R)}(R,-) \xrightarrow{\sim} \mathrm{H}^0(-).$$

Proof sketch. The strategy is to use that we know the result for $\mathbf{K}(\mathbf{Mod}_R)$. In particular, one constructs a subcategory of $\mathbf{K}(\mathbf{Mod}_R)$ which is equivalent to $\mathbf{D}(R)$, and then uses the result for $\mathbf{K}(\mathbf{Mod}_R)$.

The subcategory of $\mathbf{K}(\mathbf{Mod}_R)$ which is of interest to us is the full subcategory

$$\mathrm{KProj}_R := \{X^\bullet \in \mathbf{K}(\mathbf{Mod}_R) \mid \forall A^\bullet \text{ acyclic, } \mathrm{Hom}_{\mathbf{K}(\mathbf{Mod}_R)}(X^\bullet, A^\bullet) = 0\}$$

where we recall that $A^\bullet$ is acyclic if $\mathrm{H}^i(A^\bullet) = 0$ for all $i \in \mathbb{Z}$. Note that $R \in \mathrm{KProj}_R$. The composition

$$\mathrm{KProj}_R \to \mathbf{K}(\mathbf{Mod}_R) \to \mathbf{D}(R)$$

is an equivalence (see [Kra07, 5.1 & 5.2]) with quasi-inverse $p\colon \mathbf{D}(R) \xrightarrow{\sim} \mathrm{KProj}_R$, and thus for any $X^\bullet \in \mathbf{D}(R)$ we can compute

$$\mathrm{Hom}_{\mathbf{D}(R)}(R, X^\bullet) \cong \mathrm{Hom}_{\mathrm{KProj}_R}(R, p(X^\bullet)) \cong \mathrm{Hom}_{\mathbf{K}(\mathbf{Mod}_R)}(R, p(X^\bullet)) \cong \mathrm{H}^0(p(X^\bullet)) \cong \mathrm{H}^0(X^\bullet)$$

where we note that we have $\mathrm{H}^n(X^\bullet) \cong \mathrm{H}^n(p(X^\bullet))$. □

So we can represent the cohomology functors in $\mathbf{D}(R)$ as a Hom-functor (in particular, we note that this says $\mathrm{H}^0(-)$ is corepresentable). This gives us the following neat description of the t-structure on $\mathbf{D}(R)$:

$$\mathbf{D}(R)^{\leq 0} = \{M^\bullet \in \mathbf{D}(R) \mid \forall i > 0,\, \mathrm{Hom}_{\mathbf{D}(R)}(R, M[i]^\bullet) = 0\},$$
$$\mathbf{D}(R)^{\geq 0} = \{M^\bullet \in \mathbf{D}(R) \mid \forall i < 0,\, \mathrm{Hom}_{\mathbf{D}(R)}(R, M[i]^\bullet) = 0\}.$$

The question which we now try to lead with is as follows: if we can "detect" the aisle/coisle in $\mathbf{D}(R)$ using $\mathrm{Hom}(R,-)$, can we in a more general triangulated category *define* a t-structure from being "detectable" by some fixed object? An answer to this will not come immediately, but appears in Section 6.4, and in particular in Theorem 6.21.

6.2 Compact Objects

There are several notions we will need to develop before we can prove the promised Theorem 6.21. The first of these is that of *compact objects* in a triangulated category.

Definition 6.3. Let $\mathcal{D}$ be a triangulated category. An object $G \in \mathcal{D}$ is called *compact* if, for every set-indexed collection of objects $\{Y_i\}_{i\in I}$, the morphism of Abelian groups

$$\coprod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(G, Y_i) \to \mathrm{Hom}_{\mathcal{D}}(G, \coprod_{i\in I} Y_i),$$

induced by the morphisms $\mathrm{Hom}_{\mathcal{D}}(G, Y_i) \to \mathrm{Hom}(G, \coprod_{i\in I} Y_i)$ obtained by composing with the inclusion $Y_i \hookrightarrow \coprod_{i\in I} Y_i$, is an isomorphism.

Remark 6.4. Let $\iota_i\colon Y_i \to \coprod_{i\in I} Y_i$ be the canonical inclusion. Note that the above map is given explicitly by

$$(f_i)_{i\in I} \mapsto \sum_{i\in I} \iota_i \circ f_i.$$

Remark 6.5. Why is this a property worthy of being called "compact"? Usually, compactness refers to a kind of "finiteness" property whereby one can take an infinite set and extract from it a finite set which is sufficient to get the same information. This is actually what is happening here too: to say that $G \in \mathcal{D}$ is compact is to say that for any morphism $G \to \coprod_{i\in I} Y_i$, there exists some *unique finite* subset $I_0 \subseteq I$ and a *unique* factorization

$$\begin{tikzcd}
G \arrow[rr] \arrow[dr, dashed, "\exists!"'] & & \coprod_{i\in I} Y_i. \\
& \bigoplus_{i\in I_0} Y_i \arrow[ur, hook] &
\end{tikzcd}$$

This is simply observed by knowing what coproducts in the category **Ab** of Abelian groups are. Thus, this property of being a compact object is indeed a "compactness" property.

Example 6.6. Let R be a ring. Then R is a compact object in KProj_R. We sketch an argument for this. Our strategy is to explicitly produce an inverse to the map

$$\coprod_{i\in I} \mathrm{Hom}_{\mathrm{KProj}_R}(R, X_i^\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{KProj}_R}(R, \coprod_{i\in I} X_i^\bullet).$$

Consider some map $f\colon R \to \coprod_{i\in I} X_i^\bullet$. This consists of a single map $f^0\colon R \to \coprod_{i\in I} X_i^0$, which is determined by $f^0(1) = (x_i)_{i\in I}$; in particular $f_i^0(r) = x_i r$. Note at this point that almost all of the x_i are zero, so almost all of the component maps f_i are zero, and so we have a well-defined map

$$\mathrm{Hom}_{\mathrm{KProj}_R}(R, \coprod_{i\in I} X_i^\bullet) \longrightarrow \coprod_{i\in I} \mathrm{Hom}_{\mathrm{KProj}_R}(R, X_i^\bullet)$$

given by $f \mapsto (f_i)_{i\in I}$. This is the desired inverse.

In the above definition, we require that a certain map is an isomorphism. Actually, this is slightly unnecessary: the map is always an injection (by the following result), and thus it suffices to impose surjectivity to be compact.

Proposition 6.7. *Let $\mathcal{D}$ be an additive category, and let $X \in \mathcal{D}$. Unconditionally, for any set-indexed collection $\{Y_i\}_{i\in I}$, the map*

$$\coprod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X, Y_i) \to \mathrm{Hom}_{\mathcal{D}}(X, \coprod_{i\in I} Y_i)$$

is injective.

Proof. Let $\iota_j\colon Y_j \to \coprod_{i\in I} Y_i$ be the canonical inclusion to the jth component. Let $\pi_j\colon \coprod_{i\in I} Y_i \to Y_j$ be the map defined on components by $0\colon Y_k \to Y_j$ if $k \neq j$ and id_{Y_j} if $k = j$, i.e. defined by

$$\pi_k \circ \iota_j = 0 \text{ if } k \neq j, \quad \text{and} \quad \pi_k \circ \iota_j = \mathrm{id}_{Y_j} \text{ if } k = j.$$

We will exhibit the map

$$\phi\colon \coprod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X, Y_i) \to \mathrm{Hom}_{\mathcal{D}}(X, \coprod_{i\in I} Y_i), \quad (f_i)_{i\in I} \mapsto \sum_{i\in I} \iota_i \circ f_i$$

as being part of a decomposition of the canonical monomorphism

$$\iota\colon \coprod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X, Y_i) \hookrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X, Y_i).$$

Define a map

$$\psi\colon \mathrm{Hom}_{\mathcal{D}}(X, \coprod_{i\in I} Y_i) \to \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X, Y_i)$$

on components by the maps

$$\psi_j\colon \mathrm{Hom}_{\mathcal{D}}(X, \coprod_{i\in I} Y_i) \to \mathrm{Hom}_{\mathcal{D}}(X, Y_j), \quad \phi_j(f) = \pi_j \circ f.$$

That is,

$$\psi(f) = (\pi_i \circ f)_{i\in I}.$$

Then, for any $(f_j)_{j\in I} \in \coprod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X, Y_i)$,

$$\begin{aligned}(\psi \circ \phi)((f_j)_{j\in I}) = \psi\left(\sum_{j\in I} \iota_j \circ f_j\right) &= \left(\sum_{j\in I} \pi_i \circ \iota_j \circ f_j\right)_{i\in I} \\ &= (\iota_i \circ f_i)_{i\in I} = \iota((f_i)_{i\in I})\end{aligned}$$

so that $\iota = \psi \circ \phi$. Since ι is a monomorphism, this implies that ϕ is a monomorphism, i.e. is injective. ■

Definition 6.8. Let $\mathcal{D}$ be a triangulated category admitting small coproducts. We say a class of objects $\mathcal{U}$ in $\mathcal{D}$ *generates* $\mathcal{D}$ if whenever $X \in \mathcal{D}$ satisfies $\mathrm{Hom}_{\mathcal{D}}(U[i], X) = 0$ for all $U \in \mathcal{U}$ and all $i \in \mathbb{Z}$ we have $X = 0$.

Definition 6.9. Let $\mathcal{D}$ be a triangulated category. We say $\mathcal{D}$ is *compactly generated* if there is a set $\mathcal{G}$ of compact objects of $\mathcal{D}$ such that $\mathcal{G}$ generates $\mathcal{D}$.

The reason we are interested in these notions is because we will later, in the process of proving Theorem 6.21, want to apply the following proposition.

Proposition 6.10. *Let $\mathcal{D}$ be a triangulated category admitting small coproducts, and suppose $\mathcal{D}$ is generated by some set of objects $\mathcal{U}$ in $\mathcal{D}$. Consider a distinguished triangle*

$$X \xrightarrow{f} Y \to Z \to X[1]$$

in $\mathcal{D}$, and suppose that for all $U \in \mathcal{U}$ the induced morphisms

$$f_*\colon \mathrm{Hom}_{\mathcal{D}}(U, X) \to \mathrm{Hom}_{\mathcal{D}}(U, Y), \quad \phi \mapsto f \circ \phi$$

are isomorphisms.

(i) If $\mathcal{U}[1] \subseteq \mathcal{U}$, then for all $U \in \mathcal{U}$ and $i > 0$ we have

$$\mathrm{Hom}_{\mathcal{D}}(U[i], Z) = 0.$$

(ii) If $\mathcal{U}[1] = \mathcal{U}$, then $Z = 0$ and f is an isomorphism.

Proof. (i) Let $U \in \mathcal{U}$, $i > 0$, and let $g : Y \to Z$ and $h : Z \to X[1]$ be the morphisms in the distinguished triangle above. By applying $\mathrm{Hom}(U[i], -)$, we get the exact sequence

$$\mathrm{Hom}(U[i], X) \xrightarrow{f_*} \mathrm{Hom}(U[i], Y) \xrightarrow{g_*} \mathrm{Hom}(U[i], Z) \xrightarrow{h_*} \mathrm{Hom}(U[i], X[1]) \xrightarrow{f[1]_*} \mathrm{Hom}(U[i], Y[1]).$$

By assumption, f_* is an isomorphism, and furthermore, since $f[1]_*$ lies in the commutative square

$$\begin{tikzcd}
\operatorname{Hom}(U[i], X[1]) \arrow[r, "f[1]_*"] \arrow[d, "\wr"] & \operatorname{Hom}(U[i], Y[1]) \arrow[d, "\wr"] \\
\operatorname{Hom}(U[i-1], X) \arrow[r, "f_*"] & \operatorname{Hom}(U[i-1], Y)
\end{tikzcd}$$

we see that $f[1]_*$ is also an isomorphism. Therefore, the exactness of the above diagram says that $\operatorname{im} h_* = \ker f[1]_* = 0$, so $\ker h_* = \operatorname{Hom}(U[i], Z)$ and hence $\operatorname{im} g_* = \ker h_* = \operatorname{Hom}(U[i], Z)$, i.e. g_* is surjective. We then see that $\operatorname{Hom}(U[i], Z) \cong \operatorname{coker} f_*$, and since f_* is an isomorphism this is zero.

(ii) Running the above argument with i arbitrary gives that

$$\operatorname{Hom}_{\mathcal{D}}(U[i], Z) = 0 \text{ for all } U \in \mathcal{U},\ i \in \mathbb{Z}$$

and hence, since $\mathcal{U}$ generates $\mathcal{D}$, we have $Z = 0$. By Lemma 3.18, this implies f is an isomorphism. ■

Remark 6.11. Note that the requirement $\mathcal{U}[1] = \mathcal{U}$ is equivalent to $\mathcal{U}[1] \subseteq \mathcal{U}$ and $\mathcal{U}[-1] \subseteq \mathcal{U}$. Indeed, $\mathcal{U}[-1] \subseteq \mathcal{U} \iff \mathcal{U} \subseteq \mathcal{U}[1]$ and so the latter condition says exactly that $\mathcal{U}[1] \subseteq \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{U}[1]$, i.e. $\mathcal{U}[1] = \mathcal{U}$.

6.3 HOMOTOPY COLIMITS

Category theory allows us to define colimits, and in particular, colimits defined up to some canonical choice of isomorphism. An issue which arises in certain contexts is that we might want colimits which are more "loosely" defined, for example only up to some notion of *homotopy.* This problem is quite difficult to solve, and in some sense one has only fairly recently gotten closer to solving it through the various approaches to $(\infty, 1)$-categories. In the more elementary setting of triangulated categories, however, one can give a reasonably simple way to define at least certain homotopy colimits by interpreting distinguished triangles as "exact sequences up to homotopy."

Consider a category $\mathcal{C}$ and a sequence of morphisms

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \longrightarrow X_n \xrightarrow{f_n} \cdots$$

in $\mathcal{C}$. The colimit of this is an object $X \in \mathcal{C}$ equipped with maps $g_i \colon X_i \to X$ for all $i \geq 0$ such that $g_i = g_{i+1} \circ f_i$, and on top of this, X is the best possible such choice in the sense that if X' is any other object equipped with $h_i \colon X_i \to X'$ such that $h_i = h_{i+1} \circ f_i$, then there is a map $h \colon X \to X'$ such that $h_i = h \circ g_i$. One can combine this into the statement that

$$\textstyle\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\varphi} \coprod_{i=0}^{\infty} X_i \xrightarrow{(g_i)} X \longrightarrow 0$$

is exact, where (g_i) is the morphism induced by the $g_i \colon X_i \to X$, and φ is a morphism defined on components by

$$\varphi_{ij} = \begin{cases} f_j \colon X_j \to X_{j+1} & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $1 - \varphi$ is given by the matrix

$$1 - \varphi = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -f_0 & 1 & 0 & 0 & \cdots \\ 0 & -f_1 & 1 & 0 & \cdots \\ 0 & 0 & -f_2 & 1 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

To see this, note that the relation $(g_i) \circ (1 - \varphi) = 0$ says exactly that

$$\begin{pmatrix} g_0 & g_1 & g_2 & g_3 \cdots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -f_0 & 1 & 0 & 0 & \cdots \\ 0 & -f_1 & 1 & 0 & \cdots \\ 0 & 0 & -f_2 & 1 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} g_0 - g_1 \circ f_0 & g_1 - g_2 \circ f_1 & \cdots \end{pmatrix} = 0$$

so that $g_i = g_{i+1} \circ f_i$. The universality is then the fact that X is the cokernel of $1 - \varphi$ via (g_i). In particular, if we have an additional object X' with a map $(h_i) : \coprod_{i=0}^{\infty} X_i \to X'$ such that $(h_i) \circ (1 - \varphi) = 0$, i.e. $h_i = h_{i+1} \circ f_i$, then exactness says precisely that this induces a unique morphism $h \colon X \to X'$ such that $h \circ (g_i) = (h_i)$, i.e. such that $h \circ g_i = h_i$.

If we now replace the exact sequence with an "exact sequence up to homotopy" (see also Proposition 3.20), that is distinguished triangles, then we get the definition of a *homotopy colimit.*

Definition 6.12. Let $\mathcal{D}$ be a triangulated category admitting countable coproducts, and consider a sequence of morphisms

$$X_0 \to X_1 \to X_2 \to \cdots \to X_n \to \cdots$$

as above. A *homotopy colimit* of this sequence is an object X together with a distinguished triangle

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\varphi} \coprod_{i=0}^{\infty} X_i \longrightarrow X \longrightarrow \left(\coprod_{i=0}^{\infty} X_i\right)[1].$$

We then write $\operatorname{hocolim}_i X_i$ for any such choice of X and call it "the" homotopy colimit.

Remark 6.13. Of course, as usual in a triangulated category, the homotopy colimit is in no sense uniquely determined up to canonical isomorphism (except perhaps in exceptional circumstances). Nonetheless, just as it is useful to have notation for "the" cone of a morphism in a triangulated category, it is useful to have notation for "the" homotopy colimit as long as one remembers that what property the object should satisfy.

Homotopy colimits interact well with compact objects. In the context of triangulated categories, one defines compact objects as those whose covariant Hom-functor preserves coproducts, but in more general situations, one requires that the Hom-functor preserves all filtered colimits. If we are to believe that the homotopy colimit behaves like a colimit, then we should also expect compact objects to preserve them in a suitable sense. This is the content of the following proposition.

Proposition 6.14. *Let $\mathcal{D}$ be a triangulated category admitting countable coproducts, and consider a sequence*

$$X_0 \to X_1 \to X_2 \to \cdots \to X_n \to \cdots$$

in $\mathcal{D}$. Pick a distinguished triangle

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\varphi} \coprod_{i=0}^{\infty} X_i \longrightarrow \operatorname{hocolim}_i X_i \longrightarrow \left(\coprod_{i=0}^{\infty} X_i\right)[1]$$

defining a homotopy colimit of this sequence. Then, for any compact object $G \in \mathcal{D}$, the canonical map

$$\varinjlim_i \operatorname{Hom}_{\mathcal{D}}(G, X_i) \to \operatorname{Hom}_{\mathcal{D}}(G, \operatorname{hocolim}_i X_i)$$

is an isomorphism.

Proof. The strategy here is quite similar to the proof of Proposition 6.10. We are given a distinguished triangle involving $\operatorname{hocolim}_i X_i$ and we want to know about $\operatorname{Hom}_{\mathcal{D}}(G, \operatorname{hocolim}_i X_i)$, so we begin by applying $\operatorname{Hom}_{\mathcal{D}}(G, -)$ to get an exact sequence

$$\operatorname{Hom}(G, \coprod_i X_i) \xrightarrow{1-\varphi_*} \operatorname{Hom}(G, \coprod_i X_i) \to \operatorname{Hom}(G, \operatorname{hocolim}_i X_i) \to \operatorname{Hom}(G, \coprod_i X_i[1]) \xrightarrow{1-\varphi[1]_*} \cdots$$

and the idea is now that $1 - \varphi[1]_*$ is injective. To see this, note that by compactness we have a commutative diagram

$$\begin{tikzcd}
\coprod_i \operatorname{Hom}_{\mathcal{D}}(G, X_i) \arrow[r] \arrow[d, "\wr"] & \coprod_i \operatorname{Hom}_{\mathcal{D}}(G, X_i) \arrow[d, "\wr"] \\
\operatorname{Hom}_{\mathcal{D}}(G, \coprod_i X_i[1]) \arrow[r, "1-\varphi[1]_*"] & \operatorname{Hom}_{\mathcal{D}}(G, \coprod_i X_i[1])
\end{tikzcd}$$

where the top horizontal morphism is injective since it is given by a lower triangular matrix. As a result, $1 - \varphi[1]_*$ can be written as a composition of a number of injective maps, and hence is injective. Thus, we have an exact sequence

$$\operatorname{Hom}_{\mathcal{D}}(G, \coprod_i X_i) \xrightarrow{1-\varphi_*} \operatorname{Hom}_{\mathcal{D}}(G, \coprod_i X_i) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(G, \operatorname{hocolim}_i X_i) \longrightarrow 0.$$

Using essentially the same commutative square as above but without the shift, we obtain the exact sequence

$$\coprod_i \operatorname{Hom}_{\mathcal{D}}(G, X_i) \xrightarrow{1-\varphi_*} \coprod_i \operatorname{Hom}_{\mathcal{D}}(G, X_i) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(G, \operatorname{hocolim}_i X_i) \longrightarrow 0$$

exhibiting the relation

$$\varinjlim_i \operatorname{Hom}_{\mathcal{D}}(G, X_i) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(G, \operatorname{hocolim}_i X_i)$$

as desired. ■

Remark 6.15. Take the notation from the above proposition, and let $h_{ij} : X_i \to X_j$. It may be useful to note that $\varinjlim_i \operatorname{Hom}(G, X_i)$ can be described as follows: an element is a pair (n, f), where $n \geq 0$ and $f : G \to X_n$, modulo the equivalence relation $(n, f) \sim (m, g)$ if there is some $k \geq \max(n, m)$ such that $h_{nk} \circ f = h_{mk} \circ g$. The canonical map $\operatorname{Hom}(G, X_j) \to \varinjlim_i \operatorname{Hom}(G, X_i)$ is given by $f \mapsto (j, f)$, and the canonical map $\varinjlim_i \operatorname{Hom}(G, X_i) \to \operatorname{Hom}(G, \operatorname{hocolim}_i X_i)$ is the one induced by the maps $X_i \to \operatorname{hocolim}_i X_i$ after applying $\operatorname{Hom}(G, -)$.

The reason we have built up this machinery of homotopy colimits at all is because it gives us access to a convenient way of building certain distinguished triangles with controllable properties. As an example of something one might want, suppose we have a triangulated category which is generated by some objects. We might want to "approximate" an object in that in some way by objects built in some elementary way from the generating objects. Homotopy colimits give a method for doing this.

Specifically, if we are in the situation of a triangulated category with some generators, then homotopy colimits give us a way to approximate any object of the triangulated category by a sequence of objects obtained as extensions of coproducts of the generators, with the comparison between the "real thing" and the "approximation" given by a morphism from the homotopy colimit. Furthermore, when the generating objects are compact, this operation loses reasonably little information. This is encapsulated in the following theorem.

Theorem 6.16. *Let $\mathcal{D}$ be a triangulated category admitting small coproducts which is generated by some set $\mathcal{G}$ of (not necessarily compact!) objects, and let $X \in \mathcal{D}$. Then there exists a sequence*

$$X_0 \xrightarrow{h_0} X_1 \xrightarrow{h_1} X_2 \xrightarrow{h_2} \cdots \longrightarrow X_n \xrightarrow{h_n} \cdots$$

of objects of $\mathcal{D}$, together with morphisms $x_i \colon X_i \to X$, such that

(a) X_0 is the coproduct of some objects in $\mathcal{G}$,

(b) for every $i \geq 0$, there is an object $Y_i \in \mathcal{D}$, which is a coproduct of objects in $\mathcal{G}$, and a distinguished triangle
$$Y_i \to X_i \to X_{i+1} \to Y_i[1],$$

(c) for all $i \geq 0$, $x_i = x_{i+1} \circ h_i$,

(d) letting $(p_i) : \coprod_i X_i \to \operatorname{hocolim}_i X_i$ be the morphism in the definition of the homotopy colimit, there exists a morphism $u \colon \operatorname{hocolim}_i X_i \to X$ such that $(x_i) = u \circ (p_i)$,

(e) for any $G \in \mathcal{G}$, the induced morphism $u_* : \operatorname{Hom}_{\mathcal{D}}(G, \operatorname{hocolim}_i X_i) \to \operatorname{Hom}_{\mathcal{D}}(G, X)$ is surjective, and

(f) if the objects of $\mathcal{G}$ are compact, so that $\mathcal{D}$ is compactly generated, then for all $G \in \mathcal{G}$, the induced morphism $u_* \colon \operatorname{Hom}_{\mathcal{D}}(G, \operatorname{hocolim}_i X_i) \to \operatorname{Hom}_{\mathcal{D}}(G, X)$ is an isomorphism.

Proof. The proof essentially consists of two parts: constructing the desired sequence by induction, and showing that one has an isomorphism with the homotopy colimit.

The first part, construction, begins by constructing X_0. Since we have small coproducts $\mathcal{D}$, we may construct the somewhat ridiculous coproduct

$$X_0 := \coprod_{G \in \mathcal{G}} \ \coprod_{f \in \operatorname{Hom}_{\mathcal{D}}(G,X)} G$$

and the tautological morphism $x_0 \colon X_0 \to X$, given on the (G, f)th component by $f : G \to X$ itself. This has the obvious property that for any $G \in \mathcal{G}$, the map

$$x_{0,*} \colon \operatorname{Hom}_{\mathcal{D}}(G, X_0) \to \operatorname{Hom}_{\mathcal{D}}(G, X)$$

is surjective. In particular, if we have a map $f \colon G \to X$ then this is, by definition of X_0, given by $x_0 \circ \iota_{G,f}$, where $\iota_{G,f} \colon G \to X_0$ is the canonical inclusion to the (G, f)th component.

Suppose we have constructed X_i and $x_i : X_i \to X$ for some $i \geq 0$. To construct Y_i, we consider the coproduct

$$Y_i := \coprod_{G \in \mathcal{G}} \ \coprod_{f \in \ker(x_{i,*})} G$$

and the associated canonical map $y_i : Y_i \to X_i$ given on (G, f)th component by $f : G \to X_i$. This defines Y_i and a map $Y_i \to X_i$.

Now we explain how, given X_i, Y_i, and the map $y_i : Y_i \to X_i$, one constructs the data for the $(i + 1)$th step. We obtain X_{i+1} as the cone of y_i. In particular, we choose a distinguished triangle

$$Y_i \xrightarrow{y_i} X_i \xrightarrow{h_i} X_{i+1} \longrightarrow Y_i[1].$$

From the definition of Y_i and the map y_i, we see that for any $G \in \mathcal{G}$ and $f \in \ker x_{i,*}$, we have that the diagram

$$\begin{tikzcd}
 & G \arrow[dl, "\iota_{G,f}"'] \arrow[d, "f"] & & \\
Y_i \arrow[r, "y_i"] \arrow[dr, "0"'] & X_i \arrow[r, "h_i"] \arrow[d, "x_i"] & X_{i+1} \arrow[r] & Y_i[1] \\
 & X & &
\end{tikzcd}$$

commutes since $x_i \circ y_i \circ \iota_{G,f} = x_i \circ f = 0$ (in turn because $f \in \ker x_{i,*}$). Here, the particular choice of G and f was arbitrary, so we see that $x_i \circ y_i$ is zero on every component, i.e. $x_i \circ y_i = 0$. Thus, by the weak cokernel property of the cone (see Proposition 3.20), there exists some (not necessarily unique!) morphism $x_{i+1} \colon X_{i+1} \to X$ such that

$$\begin{tikzcd}
Y_i \arrow[r, "y_i"] \arrow[dr, "0"'] & X_i \arrow[r, "h_i"] \arrow[d, "x_i"] & X_{i+1} \arrow[r] \arrow[dl, "x_{i+1}"] & Y_i[1] \\
 & X & &
\end{tikzcd}$$

commutes. This constructs X_{i+1} and x_{i+1} as desired.

Thus, by induction, we produce a diagram

$$\begin{tikzcd}
X_0 \arrow[r, "h_0"] \arrow[d, "x_0"] & X_1 \arrow[r, "h_1"] \arrow[dl, "x_1"] & \cdots \arrow[r] & X_n \arrow[r, "h_n"] \arrow[dlll, bend left, "x_n"] & \cdots \\
X & & & &
\end{tikzcd}$$

such that properties (a), (b), and (c) of the theorem statement are satisfied.

To get a morphism as in (d), first choose a distinguished triangle

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\varphi} \coprod_{i=0}^{\infty} X_i \xrightarrow{(p_i)} \operatorname{hocolim}_i X_i \longrightarrow \left(\coprod_{i=0}^{\infty} X_i\right)[1].$$

Note then that $(x_i) \circ (1 - \varphi) = 0$, since by definition we have $x_i = x_{i+1} \circ h_i$. Therefore, again by the weak cokernel property of cones, we obtain a (non-unique!) morphism $u \colon \operatorname{hocolim}_i X_i \to X$ such that

$$\begin{tikzcd}
\coprod_i X_i \arrow[r, "1-\varphi"] \arrow[dr, "0"'] & \coprod_i X_i \arrow[r, "(p_i)"] \arrow[d, "(x_i)"] & \operatorname{hocolim}_i X_i \arrow[r] \arrow[dl, "u"] & \left(\coprod_i X_i\right)[1] \\
 & X & &
\end{tikzcd}$$

commutes. We now prove (e), that for every $G \in \mathcal{G}$ the morphism

$$u_* \colon \operatorname{Hom}(G, \operatorname{hocolim}_i X_i) \to \operatorname{Hom}(G, X)$$

is surjective. By the definition of u, we see that, letting $\iota_0 : X_0 \to \coprod_i X_i$ be the canonical inclusion,

$$u \circ p_0 = u \circ (p_i) \circ \iota_0 = (x_i) \circ \iota_0 = x_0.$$

Applying $\operatorname{Hom}_{\mathcal{D}}(G, -)$, this implies that the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{D}}(G, X_0) \arrow[rr, "p_{0,*}"] \arrow[dr, "x_{0,*}"'] & & \operatorname{Hom}_{\mathcal{D}}(G, \operatorname{hocolim}_i X_i) \arrow[dl, "u_*"] \\
 & \operatorname{Hom}_{\mathcal{D}}(G, X) &
\end{tikzcd}$$

commutes. As observed earlier during the construction of X_0, $x_{0,*}$ is surjective. Thus, since $x_{0,*} = u_* \circ p_{0,*}$, we see that u_* is surjective. This proves (e)

Finally, to prove (f), suppose that the objects of $\mathcal{G}$ are compact. We will now prove that the above map $u_* \colon \mathrm{Hom}(G, \mathrm{hocolim}_i X_i) \to \mathrm{Hom}(G, X)$ is injective. To this end, suppose that we have some $f \in \mathrm{Hom}_{\mathcal{D}}(G, \mathrm{hocolim}_i X_i)$ such that $u_*(f) = u \circ f = 0$. Since $G \in \mathcal{G}$ is compact, Proposition 6.14 gives us that the canonical morphism

$$\varinjlim_i \mathrm{Hom}(G, X_i) \to \mathrm{Hom}(G, \mathrm{hocolim}_i X_i)$$

is an isomorphism. Taking the inverse image of f, we see that there is some $n \geq 0$ and some $g \colon G \to X_n$ such that $f = p_n \circ g$ (see the end of Remark 6.15). We then have a commutative diagram

$$\begin{tikzcd}
 & & X_n \arrow[dl, "p_n"'] \arrow[dd, "x_n"] \\
G \arrow[urr, bend left, "g"] \arrow[r, "f"] \arrow[drr, bend right, "0"'] & \mathrm{hocolim}_i X_i \arrow[dr, "u"] & \\
 & & X
\end{tikzcd}$$

which displays that $x_n \circ g = 0$. Explicitly, we can compute

$$x_n \circ g = u \circ p_n \circ g = u \circ f = 0$$

so that $g \in \ker x_{n,*}$. The last step is to plug this into the data from (b). In particular, we have the commutative diagram

$$\begin{tikzcd}
 & G \arrow[dl, "\iota_{G,g}"'] \arrow[d, "g"] & & \\
Y_n \arrow[r, "y_n"] & X_n \arrow[r, "h_n"] \arrow[d, "p_n"] & X_{n+1} \arrow[r] \arrow[dl, "p_{n+1}"] & Y_n[1] \\
 & \mathrm{hocolim}_i X_i & &
\end{tikzcd}$$

which allows us to compute

$$f = p_n \circ g = p_{n+1} \circ (h_n \circ y_n) \circ \iota_{G,g} = p_{n+1} \circ 0 \circ \iota_{G,g} = 0$$

and we arrive at the conclusion that $f = 0$, so that u_* is injective. ■

Here is a fun and immediate corollary.

Corollary 6.17. *Let $\mathcal{D}$ be a compactly generated triangulated category, and let $\mathcal{G}$ be any generating set of compact objects. Then the smallest triangulated category in $\mathcal{D}$ closed under small coproducts which contains $\mathcal{G}$ is $\mathcal{D}$ itself.*

Proof. Apply Theorem 6.16 along with Proposition 6.10. In particular, replace $\mathcal{G}$ by the set

$$\mathcal{G}' := \{G[i] \mid G \in \mathcal{G},\ i \in \mathbb{Z}\},$$

which we can do since these are all compact still (where we note that shifts of compact objects are compact), and we need to be closed under shifts anyway (so this operation $\mathcal{G} \mapsto \mathcal{G}'$ doesn't change the results). ■

Later (Proposition 6.24), we will see a similar result proved in basically the same way play a role in the proof of the central Theorem 6.21.

6.4 SILTING OBJECTS

In the motivating example of $\mathbf{D}(R)$, we saw that an important property of the standard t-structure on this category was that one can detect if a complex $X^\bullet \in \mathbf{D}(R)$ is in either $\mathbf{D}(R)^{\leq 0}$ or $\mathbf{D}(R)^{\geq 0}$ using the embedding of R into $\mathbf{D}(R)$. In particular, one can compute the ith cohomology of $X^\bullet$ by computing the Hom-set $\operatorname{Hom}(R, X^\bullet[i])$, which allowed us to get a description of the t-structure in terms of such Hom-sets.

The idea in generalizing this is: if we can use R to detect these things in $\mathbf{D}(R)$, is it possible to *define* a t-structure by choosing an object S in $\mathcal{D}$ and forming the aisle/coisle by requiring that $\operatorname{Hom}_{\mathcal{D}}(S, X[i]) = 0$ for appropriate choices of i? In general, the answer is no. However, perhaps it is true if we impose additional conditions upon the object S, and one may wonder what such conditions should be. In the example we know of, we pick out the following two properties: R is a *compact generator* of $\mathbf{D}(R)$, and $\operatorname{Hom}(R, R[i]) = 0$ for all $i \neq 0$. Actually, we will weaken the latter requirement, but in any case, these conditions are essentially what motivate the definition of a silting object.

Definition 6.18. Let $\mathcal{D}$ be a triangulated category, and let $S \in \mathcal{D}$. Define the full subcategories $\mathcal{D}_S^{\leq 0}$ and $\mathcal{D}_S^{\geq 0}$ by

$$\mathcal{D}_S^{\leq 0} := \{X \in \mathcal{D} \mid \forall i > 0,\ \operatorname{Hom}_{\mathcal{D}}(S, X[i]) = 0\},$$
$$\mathcal{D}_S^{\geq 0} := \{X \in \mathcal{D} \mid \forall i < 0,\ \operatorname{Hom}_{\mathcal{D}}(S, X[i]) = 0\}.$$

We say S is a *silting object* if S is a compact generator of $\mathcal{D}$ and $S \in \mathcal{D}_S^{\leq 0}$, i.e. if S satisfies

$$\forall i > 0, \quad \operatorname{Hom}_{\mathcal{D}}(S, S[i]) = 0.$$

Remark 6.19. One says S is *tilting* it $S \in \mathcal{D}_S^{\leq 0}$ and $S \in \mathcal{D}_S^{\geq 0}$. Thus, R is actually a *tilting* object of $\mathbf{D}(R)$, not just a silting object.

Remark 6.20. Silting objects originated in the a paper of Keller & Vossieck [KV88] from 1988 on investigating aisles in derived categories, based on the historically more senior concept of tilting objects. After this, however, the concept was largely dropped from the mathematical consciousness, with relatively few papers dedicated to it for many years. In 2002, Hoshino, Kato, & Miyachi [HKM02] used the concept "indirectly" in that they simply spelled out the assumptions without referring directly to "silting objects," and they were the first to notice that silting objects induce t-structures. The below Theorem 6.21 is essentially [HKM02, Thm. 1.3]. After the paper of Hoshino–Kato–Miyachi, there was again relatively little activity on this topic until ten years later, when Aihara & Iyama in [AI12] managed to relate it to the already fairly prominent concept of "mutation," thus putting it in the context of, and using it to better explain, an existing theory. Since then, there has been progress on non-compact versions of silting objects in, for example, [PV17].

There are two central results of interest to us regarding silting objects, both due to Hoshino, Kato, & Miyachi. We encapsulate them both in the following theorem statement:

Theorem 6.21 (Hoshino–Kato–Miyachi)**.** *Let $\mathcal{D}$ be a triangulated category, and let $S \in \mathcal{D}$ be a silting object. Then*

(i) the pair $(\mathcal{D}_S^{\leq 0}, \mathcal{D}_S^{\geq 0})$ forms a t-structure on $\mathcal{D}$, and

(ii) letting $\mathcal{D}_S^{\heartsuit}$ be the heart of this t-structure, $\operatorname{Hom}_{\mathcal{D}}(S,-)$ *gives an equivalence*

$$\operatorname{Hom}_{\mathcal{D}}(S,-)\colon \mathcal{D}_S^{\heartsuit} \xrightarrow{\sim} \mathbf{Mod}_{\operatorname{End}_{\mathcal{D}}(S)}$$

between $\mathcal{D}_S^{\heartsuit}$ and the category of right modules over the endomorphism ring of S.

The proof of this theorem will involve quite a few steps. We will break it up into several lemmas and propositions.

Lemma 6.22. *Let $\mathcal{D}$ be a triangulated category and let $S \in \mathcal{D}$ be a silting object. Then*

(i) $\mathcal{D}_S^{\leq 0}[1] \subseteq \mathcal{D}_S^{\leq 0}$ and $\mathcal{D}_S^{\geq 0}[-1] \subseteq \mathcal{D}_S^{\geq 0}$, and

(ii) $\mathcal{D}_S^{\leq 0}$ and $\mathcal{D}_S^{\geq 0}$ are closed under small products and coproducts.

Proof. (i) This is immediate from the definition, and in fact does not depend on S be a silting object. Indeed, if $\operatorname{Hom}_{\mathcal{D}}(S, X[i]) = 0$ for all $i > 0$, then certainly $\operatorname{Hom}_{\mathcal{D}}(S, X[i+1]) = 0$ for all $i > 0$.

(ii) Let I be some indexing set and let $\{X_i\}_{i\in I}$ be a collection of objects of $\mathcal{D}$. Then, by the definition of the product and since S is compact, we have the following two natural isomorphisms:

$$\operatorname{Hom}_{\mathcal{D}}(S, \prod_{i\in I} X_i) \xrightarrow{\sim} \prod_{i\in I} \operatorname{Hom}_{\mathcal{D}}(S, X_i), \quad \coprod_{i\in I} \operatorname{Hom}_{\mathcal{D}}(S, X_i) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(S, \coprod_{i\in I} X_i).$$

Therefore, if we have $X_i \in \mathcal{D}_S^{\leq 0}$ for all $i \in I$, then it is clear that $\prod_{i\in I} X_i \in \mathcal{D}_S^{\leq 0}$ and $\coprod_{i\in I} X_i \in \mathcal{D}_S^{\leq 0}$, along with the dual statements for $\mathcal{D}_S^{\geq 0}$. ■

Remark 6.23. Note that this implies that $\mathcal{D}_S^{\heartsuit}$ is closed under small products and coproducts.

Proposition 6.24. *Let $\mathcal{D}$ be a triangulated category and let $S \in \mathcal{D}$ be a silting object. Then $\mathcal{D}_S^{\leq 0}$ is the smallest full subcategory of $\mathcal{D}$ which contains S and is closed under extensions, small coproducts, and non-negative shifts.*

Proof. Suppose we have a distinguished triangle

$$X \to Y \to Z \to X[1]$$

where $X, Z \in \mathcal{D}_S^{\leq 0}$. Applying $\operatorname{Hom}(S,-)$ yields the exact sequence

$$\operatorname{Hom}(S, X[i]) \to \operatorname{Hom}(S, Y[i]) \to \operatorname{Hom}(S, Z[i])$$

for every $i > 0$, where we note that $\operatorname{Hom}(S, X[i]) = \operatorname{Hom}(S, Z[i]) = 0$ by definition of $\mathcal{D}_S^{\leq 0}$. Thus, $\operatorname{Hom}(S, Y[i]) = 0$ for all $i > 0$, so $Y \in \mathcal{D}_S^{\leq 0}$.

For the converse, we will apply Theorem 6.16 and Proposition 6.10. In particular, let $X \in \mathcal{D}_S^{\leq 0}$, and define

$$\mathcal{G} := \{S[i] \mid i \geq 0\}.$$

This is a set of compact objects generating $\mathcal{D}$ (since S generates $\mathcal{D}$), and thus we may apply the theorem. The theorem produces a sequence of objects and morphisms

$$X_0 \to X_1 \to X_2 \to \cdots$$

where we have a very nice morphism $u\colon \operatorname{hocolim}_i X_i \to X$, and where each X_i further sits in a distinguished triangle

$$Y_i \to X_i \to X_{i+1} \to Y_i[1]$$

where Y_i is a coproduct of objects in $\mathcal{G}$. This implies that all the X_i are formed by non-negative shifts, coproducts, and extensions of objects in $\mathcal{G}$. Since we have a distinguished triangle

$$\coprod_i X_i \xrightarrow{1-\varphi} \coprod_i X_i \longrightarrow \operatorname{hocolim}_i X_i \to (\coprod_i X_i)[1]$$

we see that also $\operatorname{hocolim}_i X_i$ is formed by such a procedure, and thus it suffices to show that u is an isomorphism to prove the result.

Consider a distinguished triangle

$$\operatorname{hocolim}_i X_i \xrightarrow{u} X \to Z \to (\operatorname{hocolim}_i X_i)[1]$$

We want to show that $\operatorname{Hom}(S[i], Z) = 0$ for all $i \in \mathbb{Z}$, since because S generates $\mathcal{D}$ this implies that $Z = 0$. Note that by the first half of this proof, together with Lemma 6.22, the smallest full subcategory containing S which is closed under non-negative shifts, small products & coproducts, and extensions is by default a full subcategory of $\mathcal{D}_S^{\leq 0}$. This implies that $\operatorname{hocolim}_i X_i \in \mathcal{D}_S^{\leq 0}$. The above distinguished triangle thus exhibits Z as an extension of two objects in $\mathcal{D}_S^{\leq 0}$, and therefore we see that $Z \in \mathcal{D}_S^{\leq 0}$. Therefore, we automatically have that

$$\operatorname{Hom}_{\mathcal{D}}(S, Z[i]) = 0 \text{ for all } i > 0.$$

By (f) in Theorem 6.16, the morphism $u\colon \operatorname{hocolim}_i X_i \to X$ has the property that

$$u_*\colon \operatorname{Hom}_{\mathcal{D}}(S[i], \operatorname{hocolim}_i X_i) \to \operatorname{Hom}_{\mathcal{D}}(S[i], X)$$

is an isomorphism for every $i \geq 0$. Therefore, Proposition 6.10 implies that

$$\forall j > 0, \quad \operatorname{Hom}_{\mathcal{D}}(S, Z[-j]) \cong \operatorname{Hom}_{\mathcal{D}}(S[j], Z) = 0$$

Thus we know that $\operatorname{Hom}_{\mathcal{D}}(S[i], Z) = 0$ for all $i \neq 0$. The case $i = 0$ is easy: applying $\operatorname{Hom}(S, -)$ to the above distinguished triangle gives the exact sequence

$$\operatorname{Hom}_{\mathcal{D}}(S, \operatorname{hocolim}_i X_i) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(S, X) \to \operatorname{Hom}_{\mathcal{D}}(S, Z) \to \operatorname{Hom}_{\mathcal{D}}(S, (\operatorname{hocolim}_i X_i)[1]) = 0$$

and therefore $\operatorname{Hom}_{\mathcal{D}}(S, Z)$ is the cokernel of an isomorphism, i.e. it is zero. Thus, we conclude that $Z = 0$, so Lemma 3.18 says u is an isomorphism. ■

The above are essentially enough in order to prove statement (i) of Theorem 6.21. To prove statement (ii), we still need three more lemmas, two of them purely statements about Abelian categories, and the last a result about endomorphisms of silting objects. We will state and prove the last one, Lemma 6.27, at the very end.

Lemma 6.25. *Let $\mathcal{A}$ be an Abelian category admitting small coproducts, and let $P \in \mathcal{A}$ be a projective object. Suppose $\mathcal{A}$ satisfies the following two conditions:*

(a) For all objects $X \in \mathcal{A}$, there exists a set I and an epimorphism

$$\coprod_{i \in I} P \twoheadrightarrow X.$$

(b) (P is compact.) For any set I and I-indexed family $\{X_i\}_{i \in I}$ of objects in $\mathcal{A}$ the canonical morphism

$$\coprod_{i \in I} \operatorname{Hom}_{\mathcal{A}}(P, X_i) \to \operatorname{Hom}_{\mathcal{A}}(X, \coprod_{i \in I} X_i)$$

is an isomorphism.

Then the functor $\operatorname{Hom}_{\mathcal{A}}(P,-)$ *gives an additive equivalence*

$$\operatorname{Hom}_{\mathcal{A}}(P,-)\colon \mathcal{A} \xrightarrow{\sim} \mathbf{Mod}_{\operatorname{End}(P)}.$$

Proof. First, to be explicit, the (right) module structure is given by composition. That is, for an endomorphism $p\colon P \to P$, the action of this on some $f\colon P \to X$ is $fp = f \circ p$. This clearly turns $\operatorname{Hom}_{\mathcal{A}}(P,-)$ into an additive functor of the above type.

We now show $\operatorname{Hom}_{\mathcal{A}}(P,-)$ is fully faithful. Suppose that we have $\varphi : X \to Y$ and that $\varphi_* = 0\colon \operatorname{Hom}(P,X) \to \operatorname{Hom}(P,Y)$. We want to use the Yoneda lemma to deduce that $\varphi = 0$, and to do this we use assumption (a). Let $Z \in \mathcal{A}$, and find some set I with an epimorphism $\pi\colon \coprod_{i\in I} P \twoheadrightarrow Z$. Then we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}(Z,X) \arrow[r, "\varphi_*"] \arrow[d, hook, "\pi^*"] & \operatorname{Hom}_{\mathcal{A}}(Z,Y) \arrow[d, hook, "\pi^*"] \\
\operatorname{Hom}_{\mathcal{A}}(\coprod_{i\in I} P,X) \arrow[r, "\varphi_*"] \arrow[d, "\wr"] & \operatorname{Hom}_{\mathcal{A}}(\coprod_{i\in I} P,Y) \arrow[d, "\wr"] \\
\prod_{i\in I} \operatorname{Hom}_{\mathcal{A}}(P,X) \arrow[r, "(\varphi_*)=0"] & \prod_{i\in I} \operatorname{Hom}_{\mathcal{A}}(P,Y)
\end{tikzcd}$$

where the top vertical arrows are monomorphisms since π is an epimorphism. We then see by commutativity, and since π^* is a monomorphism, that $\varphi_* = 0\colon \operatorname{Hom}_{\mathcal{A}}(Z,X) \to \operatorname{Hom}_{\mathcal{A}}(Z,Y)$ for all $Z \in \mathcal{A}$, and therefore $\varphi = 0$. Thus, $\operatorname{Hom}_{\mathcal{A}}(P,-)$ is faithful.

To see that $\operatorname{Hom}_{\mathcal{A}}(P,-)$ is full, let $X \in \mathcal{A}$ and consider an epimorphism $\coprod_{i\in I} P \twoheadrightarrow X$. Taking the kernel N of this epimorphism, we have an exact sequence

$$0 \to N \hookrightarrow \coprod_{i\in I} P \twoheadrightarrow X \to 0.$$

Since P is projective, $\operatorname{Hom}_{\mathcal{A}}(P,-)$ is exact, and therefore we have an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(P,N) \hookrightarrow \operatorname{Hom}_{\mathcal{A}}(P,\coprod_{i\in I} P) \twoheadrightarrow \operatorname{Hom}_{\mathcal{A}}(P,X) \to 0.$$

For notational simplicity, let $h(-) = h^P(-) := \operatorname{Hom}_{\mathcal{A}}(P,-)$. For any $Y \in \mathcal{A}$, we then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(X,Y) \arrow[r, hook] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(\coprod_i P,Y) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(N,Y) \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_{\operatorname{End}(P)}(h(X),h(Y)) \arrow[r, hook] & \operatorname{Hom}_{\operatorname{End}(P)}(h(\coprod_i P),h(Y)) \arrow[r] & \operatorname{Hom}_{\operatorname{End}(P)}(h(N),h(Y))
\end{tikzcd}$$

with exact rows. We already showed $h^P(-)$ was faithful, so we see that the right-most vertical arrow is a monomorphism, and furthermore, we have the isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(P,Y) = h^P(Y) \cong \operatorname{Hom}_{\operatorname{End}(P)}(h^P(P),h^P(Y))$$

since $h^P(P) = \operatorname{End}(P)$. Since $h^P(-)$ is exact, it in particular preserves colimits, and therefore we have natural isomorphisms

$$\operatorname{Hom}_{\operatorname{End}(P)}(h^P(\coprod_i P),h^P(Y)) \cong \operatorname{Hom}_{\operatorname{End}(P)}(\coprod_i h^P(P),h^P(Y)) \cong \prod_i \operatorname{Hom}_{\operatorname{End}(P)}(h^P(P),h^P(Y))$$

so that the middle arrow above is an isomorphism. By the five lemma, this implies that the map

$$\mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathrm{End}(P)}(h^P(X), h^P(Y))$$

is surjective, so that $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is full.

It remains to show that $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is essentially surjective. Let $M \in \mathbf{Mod}_{\mathrm{End}(P)}$, and choose a free resolution

$$\cdots \longrightarrow \coprod_{j\in J_1} \mathrm{End}(P) \longrightarrow \coprod_{j\in J_0} \mathrm{End}(P) \longrightarrow M \longrightarrow 0.$$

We have natural isomorphisms

$$\begin{tikzcd}
\cdots \arrow[r] & \coprod_{j\in J_1} \mathrm{End}(P) \arrow[r] \arrow[d, equal] & \coprod_{j\in J_0} \mathrm{End}(P) \arrow[r, two heads] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
\cdots \arrow[r] & \coprod_{j\in J_1} \mathrm{Hom}_{\mathcal{A}}(P,P) \arrow[r] \arrow[d, "\wr"] & \coprod_{j\in J_0} \mathrm{Hom}_{\mathcal{A}}(P,P) \arrow[r, two heads] \arrow[d, "\wr"] & M \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(P, \coprod_{j\in J_1} P) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(P, \coprod_{j\in J_0} P) \arrow[r, two heads] & M \arrow[r] & 0
\end{tikzcd}$$

and since $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is fully faithful, we obtain an exact sequence

$$\cdots \longrightarrow \coprod_{j\in J_2} P \longrightarrow \coprod_{j\in J_1} P \xrightarrow{\phi} \coprod_{j\in J_0} P.$$

Taking the cokernel of the map ϕ, we get an object $X \in \mathcal{A}$ and an exact sequence

$$\cdots \longrightarrow \coprod_{j\in J_2} P \longrightarrow \coprod_{j\in J_1} P \xrightarrow{\phi} \coprod_{j\in J_0} P \longrightarrow X \longrightarrow 0$$

which, by the exactness of $\mathrm{Hom}_{\mathcal{A}}(P,-)$ and the uniqueness of cokernels, implies that $M \cong \mathrm{Hom}_{\mathcal{A}}(P,X)$. This concludes the proof. ■

The second lemma we need is one which helps us apply the preceding one.

Lemma 6.26. *Let $\mathcal{A}$ be an Abelian category admitting small coproducts, and let $P \in \mathcal{A}$ be a projective object. Then the following are equivalent:*

(i) For all $X \in \mathcal{A}$, there exists a set I and an epimorphism

$$\coprod_{i\in I} P \twoheadrightarrow X.$$

(ii) For all $X \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(P,X) = 0$ implies $X = 0$.

Proof. (i) $\implies$ (ii). Let $X \in \mathcal{D}$ and suppose $\mathrm{Hom}_{\mathcal{A}}(P,X) = 0$. By assumption we have a set I and an epimorphism

$$p\colon \coprod_{i\in I} P \twoheadrightarrow X$$

so we may compute

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} P, X) \cong \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(P,X) \cong 0$$

so that $p = 0$. We then have $0 = \mathrm{id}_X \circ p = 0 \circ p$, and therefore p being an epimorphism implies $\mathrm{id}_X = 0$, so $X = 0$.

(ii) $\implies$ (i). Let $X \in \mathcal{D}$. We will construct an epimorphism of the desired form by considering the simplest possible such candidate morphism, namely

$$p\colon \coprod_{f\in\mathrm{Hom}_{\mathcal{A}}(P,X)} P \to X$$

defined on the fth component exactly by the morphism $f\colon P \to X$, i.e. $p_f = f$. We prove that the cokernel of p is zero, with the strategy being to use the assumption (ii). Let $Y = \operatorname{coker} p$, and note that we then have an exact sequence

$$\textstyle\coprod_{f\in\mathrm{Hom}(P,X)} P \xrightarrow{p} X \twoheadrightarrow Y \longrightarrow 0.$$

If we have a morphism $h : P \to Y$, then this lifts to a morphism $g : P \to X$ such that h is the composition $P \xrightarrow{g} X \twoheadrightarrow Y$. However, by the definition of p, this itself factors through the inclusion $\iota_g\colon P \to \coprod_{f\in\mathrm{Hom}(P,X)} P$, so we have the diagram

$$\begin{tikzcd}
 & & & P \arrow[d, "h"] \arrow[lll, dashed, bend right, "\iota_g"'] \arrow[ll, dashed, bend right, "g"'] & \\
\coprod_{f\in\mathrm{Hom}(P,X)} P \arrow[r, "p"] & X \arrow[rr, two heads] & & Y \arrow[r] & 0.
\end{tikzcd}$$

But then we see that h is the composition

$$(P \xrightarrow{\iota_g} \coprod_f P \xrightarrow{p} X \twoheadrightarrow Y) = P \xrightarrow{0} Y$$

and thus $h = 0$. This shows that $\mathrm{Hom}_{\mathcal{A}}(P, Y) = 0$, so $Y = 0$ and hence p is an epimorphism. ■

We now combine all of the above work into a proof.

Proof of Theorem 6.21. (i) By Lemma 6.22, we have (T2). To prove (T1), let $Y \in \mathcal{D}_S^{\geq 0}$ and consider the full subcategory $\mathcal{U}_Y$ of $\mathcal{D}$ spanned by those objects U such that

$$\forall i \geq 0, \quad \mathrm{Hom}_{\mathcal{D}}(U[i], Y[-1]) = 0.$$

It is clear that this is closed under non-negative shifts, small coproducts, and extensions (by using that the Hom-functors are cohomological). Furthermore, by definition of $\mathcal{D}_S^{\geq 0}$, $S \in \mathcal{U}_Y$ since for all $i \geq 0$

$$\mathrm{Hom}_{\mathcal{D}}(S[i], Y[-1]) \cong \mathrm{Hom}_{\mathcal{D}}(S, Y[-i-1]) = 0$$

since $-i-1 < 0$. Therefore, by Proposition 6.24, we have that $\mathcal{D}_S^{\leq 0} \subseteq \mathcal{U}_Y$, which proves (T1).

To prove (T3), we observe that Theorem 6.16 produces a sequence of morphisms

$$X_0 \to X_1 \to X_2 \to \cdots$$

such that $\operatorname{hocolim}_i X_i \in \mathcal{D}_S^{\leq 0}$ (by Proposition 6.24) and where we have a morphism

$$u\colon \operatorname{hocolim}_i X_i \to X$$

such that $u_* : \mathrm{Hom}(S[j], \operatorname{hocolim}_i X_i) \to \mathrm{Hom}(S[j], X)$ is an isomorphism for all $j \geq 0$. Let Z be a cone of u, in a distinguished triangle

$$\operatorname{hocolim}_i X_i \xrightarrow{u} X \to Z \to (\operatorname{hocolim}_i X_i)[1]$$

By Proposition 6.10, we see that for all $j < 0$

$$\mathrm{Hom}_{\mathcal{D}}(S, Z[j]) = 0,$$

and following the same argument as in Proposition 6.24 we also see that $\mathrm{Hom}_{\mathcal{D}}(S, Z) = 0$. Therefore, $Z \in \mathcal{D}_S^{\geq 1}$, and the above distinguished triangle is the one demanded by (T3).

This proves that $(\mathcal{D}_S^{\leq 0}, \mathcal{D}_S^{\geq 0})$ is a t-structure.

(ii) By Theorem 5.20, the heart $\mathcal{D}_S^{\heartsuit} = \mathcal{D}_S^{\leq 0} \cap \mathcal{D}_S^{\geq 0}$ is an Abelian category. We want to apply Lemma 6.25 to conclude the result first for the truncation $\tau^{\geq 0}S$. By Lemma 6.26, requirement (a) of the lemma follows from the assumption that S generates $\mathcal{D}$. In particular, if $X \in \mathcal{D}_S^{\heartsuit}$, then since $\mathrm{Hom}(\tau^{\geq 0}S, X) \cong \mathrm{Hom}(S, X)$, we see that

$$\mathrm{Hom}_{\mathcal{D}}(\tau^{\geq 0}S, X) = 0 \implies \mathrm{Hom}_{\mathcal{D}}(S, X) = 0 \implies X = 0.$$

Note that we use that $X \in \mathcal{D}_S^{\leq 0} \cap \mathcal{D}_S^{\geq 0}$. Requirement (b) of the lemma is simply that S is compact, which it is by the definition of being silting, together with the fact that left adjoints preserve colimits. Furthermore, by Lemma 6.22, the heart $\mathcal{D}_S^{\heartsuit}$ is closed under small coproducts, so what remains is to show that $\tau^{\geq 0}S$ is a projective object of $\mathcal{D}_S^{\heartsuit}$.

To say that $\tau^{\geq 0}S$ is projective is equivalent to saying that $\mathrm{Hom}_{\mathcal{D}_S^{\heartsuit}}(\tau^{\geq 0}S, -)$ is exact, and that in turn would follow from having $\mathrm{Ext}_{\mathcal{D}_S^{\heartsuit}}(\tau^{\geq 0}S, -) = 0$. Thus, we try to prove the latter. We use Theorem 5.30 to note that we have a natural isomorphism

$$\mathrm{Ext}_{\mathcal{D}_S^{\heartsuit}}(\tau^{\geq 0}S, X) \cong \mathrm{Hom}_{\mathcal{D}}(\tau^{\geq 0}S, X[1])$$

for all $X \in \mathcal{D}_S^{\heartsuit}$, so we must show that the right-hand side is zero. Consider the distinguished triangle

$$\tau^{\leq -1}S \longrightarrow S \longrightarrow \tau^{\geq 0}S \longrightarrow (\tau^{\leq -1}S)[1]$$

and apply the cohomological functor $\mathrm{Hom}(-, X[1])$ to get the exact sequence

$$\mathrm{Hom}((\tau^{\leq -1}S)[1], X[1]) \to \mathrm{Hom}(\tau^{\geq 0}S, X[1]) \to \mathrm{Hom}(S, X[1]) \to \mathrm{Hom}(\tau^{\leq -1}S, X[1])$$

and then note that

$$\mathrm{Hom}((\tau^{\leq -1}S)[1], X[1]) \cong \mathrm{Hom}(\tau^{\leq -1}S, X) = 0$$

since, in particular, $X \in \mathcal{D}_S^{\geq 0}$, and

$$\mathrm{Hom}(S, X[1]) = 0$$

since, in particular, $X \in \mathcal{D}_S^{\leq 0}$. Therefore, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\tau^{\geq 0}S, X[1]) \longrightarrow 0$$

so $\mathrm{Hom}_{\mathcal{D}}(\tau^{\geq 0}S, X[1]) = 0$, and we get that $\mathrm{Ext}(\tau^{\geq 0}S, X) = 0$.

Since $\tau^{\geq 0}S$ and $\mathcal{D}_S^{\heartsuit}$ now satisfy the requirements of Lemma 6.25 below, we see that we have an equivalence

$$\mathrm{Hom}_{\mathcal{D}_S^{\heartsuit}}(\tau^{\geq 0}S, -) \colon \mathcal{D}_S^{\heartsuit} \xrightarrow{\sim} \mathbf{Mod}_{\mathrm{End}(\tau^{\geq 0}S)}.$$

By Lemma 6.27 below,

$$\mathbf{Mod}_{\mathrm{End}(\tau^{\geq 0}S)} \cong \mathbf{Mod}_{\mathrm{End}(S)},$$

and since $\tau^{\geq 0}$ is left adjoint to the inclusion $\mathcal{D}_S^{\geq 0} \hookrightarrow \mathcal{D}_S$, we have that *when restricted to the heart,* there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}_S^{\heartsuit}}(\tau^{\geq 0}S, -) \cong \mathrm{Hom}_{\mathcal{D}}(S, -)$$

which completes the proof. ■

Lemma 6.27. *Let $\mathcal{D}$ be a triangulated category admitting small coproducts, and let $S \in \mathcal{D}$ be a silting object. Then the canonical map*

$$\operatorname{Hom}_{\mathcal{D}}(S, S) \to \operatorname{Hom}_{\mathcal{D}}(\tau^{\geq 0} S, \tau^{\geq 0} S)$$

is an isomorphism of algebras.

Proof. The map in question is simply $(f : S \to S) \mapsto \tau^{\geq 0} f$, and hence functoriality gives that this is a morphism of algebras (with multiplication given by composition). To see that this is an injective map, let $f \in \operatorname{End}(S)$ and suppose that $\tau^{\geq 0} f = 0$. Then, by definition of the truncation, we have a commutative diagram

$$\begin{tikzcd}
\tau^{\leq -1} S \arrow[r] & S \arrow[r] \arrow[d, "f"] & \tau^{\geq 0} S \arrow[r] \arrow[d, "0"] & (\tau^{\leq -1} S)[1] \\
\tau^{\leq -1} S \arrow[r] & S \arrow[r] & \tau^{\geq 0} S \arrow[r] & (\tau^{\leq -1} S)[1]
\end{tikzcd}$$

and thus, by the weak kernel property of cocones (see Proposition 3.20), we can find a morphism $g \colon S \to \tau^{\leq -1} S$ such that

$$\begin{tikzcd}
\tau^{\leq -1} S \arrow[r] & S \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "g"'] & \tau^{\geq 0} S \arrow[r] \arrow[d, "0"] & (\tau^{\leq -1} S)[1] \\
\tau^{\leq -1} S \arrow[r] & S \arrow[r] & \tau^{\geq 0} S \arrow[r] & (\tau^{\leq -1} S)[1]
\end{tikzcd}$$

commutes. However, since $\tau^{\leq -1} S \in \mathcal{D}_S^{\leq -1}$, we have $\operatorname{Hom}_{\mathcal{D}}(S, \tau^{\leq -1} S) = 0$, so $g = 0$, and therefore by commutativity we have $f = 0$.

To see that we have surjectivity, pick some map $h \in \operatorname{End}(\tau^{\geq 0} S)$, and note that it fits in a diagram

$$\begin{tikzcd}
\tau^{\leq -1} S \arrow[r] & S \arrow[r] & \tau^{\geq 0} S \arrow[r] \arrow[d, "h"] & (\tau^{\leq -1} S)[1] \\
\tau^{\leq -1} S \arrow[r] & S \arrow[r] & \tau^{\geq 0} S \arrow[r] & (\tau^{\leq -1} S)[1]
\end{tikzcd}$$

Since $\operatorname{Hom}_{\mathcal{D}}(S, (\tau^{\leq -1} S)[1]) = 0$, we see that the composition

$$\begin{tikzcd}
S \arrow[r] & \tau^{\geq 0} S \arrow[d, "h"] & \\
& \tau^{\geq 0} S \arrow[r] & (\tau^{\leq -1} S)[1]
\end{tikzcd}$$

must be zero, so again the weak kernel property yields a map $f \colon S \to \tau^{\leq -1} S$ such that

$$\begin{tikzcd}
\tau^{\leq -1} S \arrow[r] & S \arrow[r] \arrow[d, dashed, "f"] & \tau^{\geq 0} S \arrow[r] \arrow[d, "h"] & (\tau^{\leq -1} S)[1] \\
\tau^{\leq -1} S \arrow[r] & S \arrow[r] & \tau^{\geq 0} S \arrow[r] & (\tau^{\leq -1} S)[1]
\end{tikzcd}$$

commutes as desired. In particular, uniqueness of adjoints implies that $\tau^{\geq 0} f = h$. ■

Example 6.28. Any ring R is a silting object in $\mathbf{D}(R)$, and the t-structure generated by it is the standard t-structure. To see this, note that $\mathbf{D}(R)$ is equivalent to KProj_R, and we already saw that R is compact in KProj_R in Example 6.6. To see that it is a generator, note that if $X^\bullet \in \mathrm{KProj}_R$ is such that $\mathrm{Hom}(R, X[i]^\bullet) = 0$ for all $i \in \mathbb{Z}$, then we see that $\mathrm{H}^i(X^\bullet) = 0$ for all $i \in \mathbb{Z}$, so $X^\bullet$ is acyclic. By the definition of KProj_R, this implies that $\mathrm{Hom}(X^\bullet, X^\bullet) = 0$, so $\mathrm{id}_{X^\bullet} = 0$, which implies that $X^\bullet = 0$. Thus R is a compact generator of KProj_R, and hence a compact generator of $\mathbf{D}(R)$. That R is now a silting (and even a tilting object!) is obvious from the fact that

$$\mathrm{Hom}_{\mathbf{D}(R)}(R, R[i]) \cong \mathrm{H}^i(R)$$

and this is zero whenever $i \neq 0$.

Example 6.29. In topology, it is frequently of interest to compute the homotopy groups of some space. Usually, this is practically impossible. As a replacement, one can consider the *stable* homotopy groups, which are much more friendly for computation and also appear "in the wild." Ordinary homotopy theory essentially occurs in **hTop**, the homotopy category of topological spaces. In stable homotopy theory, one replaces topological spaces with *spectra* to obtain the *stable homotopy category* **SH**, roughly speaking obtained by "stabilizing" **hTop** under the operations of suspension and looping, in some sense linearizing it (more precisely, **SH** is a triangulated category). The stable homotopy category contains an object called the *sphere spectrum* S (obtained by repeatedly applying suspensions to a point). This is a compact generator of **SH**, and it is *connective* (meaning $\pi_i(S) = 0$ for all $i < 0$) on account of being a suspension spectrum, which in this context is exactly the condition of being silting (see, for example, [SS03, Lemma 3.5.2]).

For more information on this, one should consult the literature on stable homotopy theory, for example [SS03, 2.3.(i) & 2.3.(ii)] and the resources referenced there. For a more general result, see [SS03, Thm. 3.9.3] and possibly [HMV20, Remark 4.6]. In particular, these state that a *topological* triangulated category (i.e. one which arises as the homotopy category of certain kinds of model categories) has a (compact) silting object if and only if it is the category of module spectra over a connective ring spectrum, where the ring spectrum is then the silting object. In the above, this is given by the sphere spectrum.

6.5 NOTES ON GLUING & ENHANCEMENTS

In Section 5.6, we showed that one may glue t-structures in a recollement. This is a good illustration of a general theme: in the situation of a recollement

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] & \mathcal{D} \arrow[l, two heads, bend right] \arrow[l, two heads, bend left] \arrow[r, two heads] & \mathcal{E}. \arrow[l, hook', bend right] \arrow[l, hook, bend left] \end{tikzcd}$$

one is often interested in what structures on $\mathcal{C}$ and $\mathcal{E}$ one can transport (i.e. glue) to $\mathcal{D}$. What we have shown earlier is that it is possible for t-structures. One may then wonder if it is possible for silting objects.

Given silting objects $S \in \mathcal{C}$ and $S' \in \mathcal{E}$, it is indeed possible to "glue" them to get a silting object $S'' \in \mathcal{D}$. However, this gluing is *not* always related to the gluing of the associated t-structures. Instead, it turns out that in general silting objects are more closely related to *co-t-structures* than t-structures. We will not go into any detail on the constructions here, but an overview is that one may play the same game we played with t-structures but with co-t-structures, and obtain a similar (but also quite distinct) theory. In particular, one may glue co-t-structures in a recollement, and the gluing of silting objects is compatible with the gluing of their associated co-t-structures. In some select situations, it is possible to glue silting objects in such a way that the gluing is compatible with the induced t-structures instead. For more details on this, one can read [LVY14, §3].

Stepping away from gluing, one may ask about how silting objects behave in the context of stable ∞-categories. Very reductively, Theorem 6.21 says that a triangulated category $\mathcal{D}$ admitting coproducts and a "nice" compact generator S contains the Abelian category $\mathbf{Mod}_{\mathrm{End}(S)}$. However, it is clear that there is no reason to believe we can extend this to an equivalence between $\mathcal{D}$ and $\mathbf{D}(\mathrm{End}(S))$ in general. On the other hand, the analogue for Abelian categories is essentially just Lemma 6.25, where we see that actually any Abelian category admitting a "nice" compact generator is equivalent to a module category. Since triangulated categories are supposed to be similar to Abelian categories, we might wonder what has gone wrong (aside from conflating two different notions of "nice" compact generators).

As it turns out, if we move from triangulated categories to stable ∞-categories, the result is actually true. We give a statment of the appropriate theorem, taken directly from [Lur17] (with two slight modifications to notation, but otherwise word-for-word). Originally, it is due to Schwede and Shipley in [SS03]. We will denote the homotopy category of an ∞-category $\mathcal{C}$ by $\mathrm{h}(\mathcal{C})$. We will not explain any of the notation or terminology, and thus the statement should be read somewhat heuristically.

Theorem. [Lur17, Thm. 7.1.2.1] *Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is equivalent to $\mathbf{RMod}_R$ for some $\mathbb{E}_1$-ring R if and only if $\mathcal{C}$ is presentable and there exists a compact object $C \in \mathcal{C}$ which generates $\mathcal{C}$ in the following sense: if $D \in \mathcal{C}$ is an object having the property that $\mathrm{Ext}^n_{\mathcal{C}}(C, D) \simeq 0$ for all $n \in \mathbb{Z}$, then $D \simeq 0$.*

Remark 6.30. Lurie points out in [Lur17, Remark 7.1.2.3] that the $\mathbb{E}_1$-ring R in the above statement may be identified with $\mathrm{End}(C)$.

In the context of stable ∞-categories, as long as R is discrete, the category $\mathbf{RMod}_R$ "is" just (the enhanced version of) $\mathbf{D}(R) = \mathbf{D}(\mathbf{Mod}_R)$, and so the above theorem states that any (nice) stable ∞-category admitting a compact generator is equivalent to a module ∞-category.

Table of Selected Notation

Symbol	Meaning
$\mathcal{C}/A$	The slice category over A, i.e. the category of morphisms to A.
$\mathcal{C}^{\mathrm{op}}$	The opposite category of $\mathcal{C}$.
$\mathcal{C}_{\mathcal{S}}$	The localization of $\mathcal{C}$ at the class of morphisms $\mathcal{S}$.
$\prod$, $\coprod$	Product and coproduct.
$\mathbf{C}(\mathcal{A})$	The category of chain complexes in $\mathcal{A}$.
C_f, K_f	A cone (resp. cocone) of a morphism f in a triangulated category.
$\mathcal{D}/\mathcal{N}$	Verdier quotient of a triangulated category $\mathcal{D}$ by a null system $\mathcal{N}$.
Δ_X, ∇_X	The diagonal $X \to X \times X$ and codiagonal $X \sqcup X \to X$.
$\Delta_{X/Y}$, $\nabla_{Y/X}$	The diagonal $X \to X \times_Y X$ and codiagonal $Y \sqcup_X Y \to Y$ associated to a morphism $X \to Y$.
$\mathbf{D}(\mathcal{A})$	The derived category of the Abelian category $\mathcal{A}$.
$\mathrm{Ext}(X, Y)$	The (first) Yoneda extension group; the extensions of X by Y.
$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$	The category of functors $\mathcal{C} \to \mathcal{D}$.
$\mathrm{Fun}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$	The category of functors $\mathcal{C} \to \mathcal{D}$ sending morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{D}$.
H^n	The nth cohomology functor $\mathcal{D} \to \mathcal{D}^{\heartsuit}$ associated to a t-structure on a triangulated category $\mathcal{D}$.
$\mathrm{Hom}_{\mathcal{C}}(A, B)$	The collection of morphisms $A \to B$ in the category $\mathcal{C}$.
$\mathrm{im}\, f$	The image of the morphism f.
$\mathbf{I}(X)^{\bullet}$	The cylinder of the chain complex $X^{\bullet}$.
$\mathbf{K}(\mathcal{A})$	The homotopy category of chain complexes in $\mathcal{A}$.
KProj_R	The homotopically projective objects in $\mathbf{K}(\mathbf{Mod}_R)$; see the proof of Prop. 6.2.
$\mathbf{Mod}_R$, ${}_R\mathbf{Mod}$	The right (resp. left) modules over a ring R.
$\varprojlim$, $\varinjlim$	The limit and colimit.
$\mathrm{thick}(\mathcal{C})$	The smallest thick subcategory of a triangulated category $\mathcal{D}$ containing $\mathcal{C}$.
$\tau^{\leq n}, \tau^{\geq n}$	The truncation functors associated to a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on a triangulated category $\mathcal{D}$.
$X \xrightarrow{\sim} Y$	An isomorphism $X \to Y$.
$X \hookrightarrow Y$	An injection/monomorphism $X \to Y$.
$\mathcal{C} \hookrightarrow \mathcal{D}$	A fully faithful functor $\mathcal{C} \to \mathcal{D}$.
$X \twoheadrightarrow Y$	A surjection/epimorphism $X \to Y$.
$\mathcal{C} \twoheadrightarrow \mathcal{D}$	An essentially surjective functor $\mathcal{C} \to \mathcal{D}$.

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