Tensor Algebras and Polynomial Rings

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1 Tensor Products of Vector Spaces

Let $\Bbbk$ be a field, e.g. $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$, and let $V, V' \in \mathbf{Vect}_{\Bbbk}^{\text{f.d.}}$ be finite-dimensional $\Bbbk$-vector spaces. The tensor product $V \otimes_{\Bbbk} V'$ can be defined abstractly as a vector space satisfying a certain universal property, in that it is the unique vector space which turns bilinear maps $V \times V' \longrightarrow W$ into linear maps $V \otimes_{\Bbbk} V' \longrightarrow W$. On the other hand, this description is non-constructive, and indeed does not show that the tensor product exists.

We may give an explicit construction of the tensor product as follows:

Definition 1.1. Let V and V' be $\Bbbk$-vector spaces. Then their tensor product over $\Bbbk$ is the vector space

$$V \otimes_{\Bbbk} V' := \{ \sum_{\substack{v \in V \\ v' \in V'}} \lambda_{v,v'}(v \otimes v') \mid \text{almost all } \lambda\text{'s are } 0\}$$

subject to the relations

$$v \otimes v_1' + v \otimes v_2' = v \otimes (v_1' + v_2'), \qquad v_1 \otimes v' + v_2 \otimes v' = (v_1 + v_2) \otimes v',$$

$$\lambda(v \otimes v') = (\lambda v) \otimes v' = v \otimes (\lambda v').$$

That is, we consider the free vector space

$$\coprod_{\substack{v \in V \\ v' \in V'}} \Bbbk \cdot (v \otimes v')$$

with basis $\{v \otimes v'\}_{v \in V,\, v' \in V'}$ and then quotient by the subspace spanned by the collections

$$\{v \otimes v_1' + v \otimes v_2' - v \otimes (v_1' + v_2') \mid v \in V,\, v_1', v_2' \in V'\},$$
$$\{v_1 \otimes v' + v_2 \otimes v' - (v_1 + v_2) \otimes v' \mid v_1, v_2 \in V,\, v' \in V'\},$$
$$\{\lambda(v \otimes v') - (\lambda v) \otimes v' \mid \lambda \in \Bbbk,\, v \in V,\, v' \in V'\},$$
$$\{\lambda(v \otimes v') - v \otimes (\lambda v') \mid \lambda \in \Bbbk,\, v \in V,\, v' \in V'\}.$$

In other words, the tensor product $V \otimes_{\Bbbk} V'$ is the vector space generated by the symbols $v \otimes v'$ under addition, scaling, and the given relations. This is still quite hard to understand, because it is a large amount of data, but luckily if we assume that V and V' are finite-dimensional then we have a much more pleasant description.

Proposition 1.2. *Let $V, V' \in \mathbf{Vect}_{\Bbbk}^{\text{f.d.}}$. Suppose V has a basis $\{e_i\}_{i=1}^m$ and suppose that V' has a basis $\{e_j'\}_{j=1}^n$. Then $V \otimes_{\Bbbk} V'$ has a basis given by $\{e_i \otimes e_j'\}_{1 \le i \le m,\, 1 \le j \le n}$.*

Proof. Consider a basic element $v \otimes v'$ in $V \otimes_{\Bbbk} V'$. Since we have bases of V and V', we can write

$$v = \sum_i \lambda_i e_i \quad \text{and} \quad v' = \sum_j \mu_j e'_j.$$

But then

$$\begin{aligned} v \otimes v' = \left(\sum_i \lambda_i e_i\right) \otimes v' = \sum_i \lambda_i (e_i \otimes v') = \sum_i \left(\lambda_i e_i \otimes \left(\sum_j \mu_j e'_j\right)\right) \\ = \sum_i \left(\sum_j \lambda_i e_i \otimes \mu_j e'_j\right) = \sum_{i,j} \lambda_i \mu_j (e_i \otimes e'_j). \end{aligned}$$

Since $V \otimes_{\Bbbk} V'$ is generated by elements of the form $v \otimes v'$, and elements of the latter form are generated by elements of the form $e_i \otimes e'_j$, we see that the set $\{e_i \otimes e'_j\}_{i,j}$ spans $V \otimes_{\Bbbk} V'$.

Now observe that since $V \otimes_{\Bbbk} V'$ is spanned by a finite set, it is necessarily finite-dimensional. It is then a standard fact that

$$\dim_{\Bbbk}(V \otimes_{\Bbbk} V') = \dim_{\Bbbk} \mathrm{Hom}_{\Bbbk}(V \otimes_{\Bbbk} V', \Bbbk).$$

Therefore, if we can show that the latter quantity is $\#\{e_i \otimes e'_j\}_{i,j} = m \cdot n$, then we are done.

By definition, linear maps $V \otimes_{\Bbbk} V' \longrightarrow \Bbbk$ are in linear bijection with bilinear maps $V \times V' \longrightarrow \Bbbk$. Now observe that we can define a collection $\{\phi_{i,j} : V \times V' \longrightarrow \Bbbk\}$ of such bilinear maps by bilinearly extending

$$\phi_{i,j}(e_k, e'_\ell) = \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j, \\ 0 & \text{otherwise.} \end{cases}$$

It is then clear that the collection $\{\phi_{i,j}\}$ forms a basis for the space of bilinear maps $V \times W \longrightarrow \Bbbk$, and hence that

$$\dim_{\Bbbk}(V \otimes_{\Bbbk} V') = \dim_{\Bbbk} \mathrm{Hom}_{\Bbbk}(V \otimes_{\Bbbk} V', \Bbbk) = \dim_k\{\text{bilinear maps} V \times V' \longrightarrow \Bbbk\} = m \cdot n.$$

This completes the proof. ∎

The above can simply be restated as follows:

Corollary 1.3. *Let $V \cong \Bbbk^m$ and $V' \cong \Bbbk^n$. Then this induces an isomorphism $V \otimes_{\Bbbk} V' \cong \Bbbk^{mn}$.*

2 Tensor Algebras

Now that we have access to the tensor product between vector spaces, we want to think of the symbol $\otimes$ sitting between two vectors $v \otimes v'$ as an actual "product" of some sort. The way to do this is by constructing the tensor algebra, which we do by considering "formal sums" of tensors of varying lengths. We produce two variants on this, for the convenience of the reader.

Definition 2.1. Let $V \in \mathbf{Vect}_{\Bbbk}$. The *tensor algebra* over V is

$$T(V) := \coprod_{n=0}^{\infty} V^{\otimes n} = \Bbbk \oplus V \oplus (V \otimes_{\Bbbk} V) \oplus \cdots$$

and the *large tensor algebra* over V is

$$T((V)) := \prod_{n=0}^{\infty} V^{\otimes n} = \Bbbk \times V \times (V \otimes_{\Bbbk} V) \times \cdots .$$

Remark 2.2. The difference between the two is that the first requires almost all coefficients to be zero, while the latter allows any and all sequences.

It is clear that both $T(V)$ and $T((V))$ are (∞-dimensional) $\Bbbk$-vector spaces, so all that really remains is to say what the multiplication on them is. Intuitively, we should want the multiplication to act as follows:

$$(v_1 \otimes \cdots \otimes v_m) \otimes (w_1 \otimes \cdots \otimes w_n) = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n .$$

Indeed, by bilinearity, this is enough to specify the multiplication. More explicitly, consider two arbitrary elements

$$v = \sum_d \sum_i \lambda_i (v_{i,1} \otimes \cdots \otimes v_{i,d}) \quad \text{and} \quad w = \sum_d \sum_i \mu_i (w_{i,1} \otimes \cdots \otimes w_{i,d}).$$

Then we should have

$$\begin{aligned} v \otimes w &= \left(\sum_d \sum_i \lambda_i (v_{i,1} \otimes \cdots \otimes v_{i,d}) \right) \otimes \left(\sum_d \sum_i \mu_i (w_{i,1} \otimes \cdots \otimes w_{i,d}) \right) \\ &= \sum_d \sum_i \lambda_i \left((v_{i,1} \otimes \cdots \otimes v_{i,d}) \otimes \sum_d \sum_i \mu_i (w_{i,1} \otimes \cdots \otimes w_{i,d}) \right) \\ &= \sum_d \sum_i \sum_d \sum_i \lambda_i \mu_i \left((v_{i,1} \otimes \cdots \otimes v_{i,d}) \otimes (w_{i,1} \otimes \cdots \otimes w_{i,d}) \right) \end{aligned}$$

so that bilinearity makes the definition reduce to the given one.

Example 2.3. Let V be a 1-dimensional $\Bbbk$-vector space, so that V is spanned by some vector $x \in V$. Then

$$T(V) \cong \Bbbk[x] \quad \text{and} \quad T((V)) \cong \Bbbk[\![x]\!].$$

To see this, first note that all tensor powers $V^{\otimes n}$ of V are also 1-dimensional, in particular spanned by the elements of the form

$$x \otimes \cdots \otimes x \in V^{\otimes n}.$$

To simplify notation, we write $x^{\otimes n}$ for these. After this, the second fact to note is that the element $x \in V$ generates $T(V)$ as a $\Bbbk$-algebra: trivially, any element of $T(V)$ is of the form

$$\sum_{d=0}^{n} \lambda_d x^{\otimes d} = \lambda_0 + \lambda_1 x + \lambda_2 x^{\otimes 2} + \cdots + \lambda_n x^{\otimes n}.$$

We then define a homomorphism

$$\phi \colon T(V) \longrightarrow \Bbbk[x]$$

by $\phi(x^{\otimes n}) = x^n$ and extending this linearly. Definitionally, this gives a linear isomorphism of the desired type, and it is also clear that it preserves the multiplication, hence is is actually an isomorphism $T(V) \cong \Bbbk[x]$. The computation showing that $T((V)) \cong \Bbbk[\![x]\!]$ is essentially identical; the only difference is that one has to consider an infinite sum $\sum_{d=0}^{\infty} \lambda_d x^{\otimes d}$ instead of a finite one.

Example 2.4. Let V be an n-dimensional $\Bbbk$-vector space, with basis $\{x_1, \ldots, x_n\}$. Then

$$T(V) \cong \Bbbk\langle x_1, \ldots, x_n \rangle$$

is the polynomial ring over $\Bbbk$ in n noncommutative variables. A similar description can be given for $T((V))$, being the formal power series ring over $\Bbbk$ in n noncommutative variables. The computation for this is essentially identical to Example 2.3. One sees, for example, that $T(V)$ is generated as a $\Bbbk$-algebra by the elements $\{x_1, \ldots, x_n\}$.

Intuitively, the kth piece $V^{\otimes k}$ of $T(V)$ consists of the degree k homogeneous polynomials in the x_i. For example, if $n = 2$ and $k = 3$, then a prototypical element of $V^{\otimes k}$ might be

$$2x_1^{\otimes 3} + x_1 \otimes x_2 \otimes x_1 - 3x_1 \otimes x_2^{\otimes 2}.$$

Note that when $n = 1$, there are no other variables to express any noncommutativity, so $\Bbbk[x] = \Bbbk\langle x \rangle$.